Computation of the invariant measure for a Lévy driven SDE: Rate of convergence

Fabien Panloup

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Abstract

We study the rate of convergence of some recursive procedures based on some "exact" or "approximate" Euler schemes which converge to the invariant measure of an ergodic SDE driven by a Lévy process. The main interest of this work is to compare the rates induced by "exact" and "approximate" Euler schemes. In our main result, we show that replacing the small jumps by a Brownian component in the approximate case preserves the rate induced by the exact Euler scheme for a large class of Lévy processes.

Keywords: stochastic differential equation ; Lévy process ; invariant distribution ; Euler scheme ; rate of convergence.

1 Introduction

In a recent paper (see [Pan05]), we investigated a family of several weighted empirical measures based on some Euler schemes with decreasing step in order to approximate recursively the invariant distribution ν of an ergodic jump diffusion process $X = (X_t)_{t\geq 0}$ solution to a SDE driven by a Lévy process. More precisely, let $(\bar{X}_k)_{k\geq 1}$ be such an Euler scheme with sequence of decreasing steps $(\gamma_k)_{k\geq 1}$ and let $(\eta_k)_{k\geq 1}$ be a sequence of nonnegative weights. We showed under some Lyapunov-type mean-reverting assumptions on the coefficients of the SDE and some light conditions on the steps and on the weights that,

$$\bar{\nu}_n(\omega, f) = \frac{1}{\eta_1 + \ldots + \eta_n} \sum_{k=1}^{n} \eta_k f(\bar{X}_{k-1}(\omega)) \xrightarrow{n\to+\infty} \nu(f) \qquad a.s., \tag{1}$$

for a large class of functions f including bounded continuous functions (see Proposition 1 below, or [Pan05] for more general results, *e.g.*, when ν is not unique). We obtained this result for two types of Euler schemes: the "exact" Euler scheme that is built using the true increment of the Lévy process and some "approximate" Euler schemes in which the Lévy process increments are replaced by an approximation which can be simulated.
The aim of this paper is to study the rate of *a.s.* weak convergence of $(\bar{\nu}_n)$ toward ν for these schemes and to devise some variants of our schemes which speed up this rate. This problem has been first studied, for strongly mean-reverting Brownian diffusions, by Lamberton and Pagès ([LaPa02]) when $\eta_n = \gamma_n$, and by Lemaire ([Lem06]) for more general

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weight sequences (see also [LaPa03] and [Lem05]). In particular, Lemaire established in [Lem06] that considering some more general weights does not improve the rate obtained with $\eta_n = \gamma_n$ (although some choices may improve the "sharp" rate). Following this remark and in order to limit the technical difficulties, we will focus on the case $\eta_n = \gamma_n$. However, we will assume that (X_t) solution to the Lévy driven SDE (see (2)) is a weakly mean-reverting stochastic process, *i.e.* that (X_t) satisfies a weaker Lyapunov assumption than in the previously cited papers.

As a first result, we show that the rate induced by the Exact Euler scheme (Scheme (E)) is the same as that obtained for Brownian diffusions, provided the Lévy process has moments up to order 4. In particular, the best rate is of order $n^{\frac{1}{3}}$ (see Theorems 1 and 3). However, in practice, this "exact" scheme needs the increments of the jump component of the Lévy process to be simulated in an exact way. This is not possible in general except in some particular cases (stable processes, compound Poisson process, Gamma processes,...). That is why we need to consider some approximate Euler schemes built with some approximations of the jump component, especially when the Lévy process jumps infinitely often on any compact time interval.

The canonical way to approximate the jump component is to truncate its small jumps (Scheme (P)). This amounts to replacing this jump component by a compensated compound Poisson process (CCPP). For this type of approximation, the smaller the truncation threshold is, the closer the law of the corresponding CCPP is to that of the true jump component, but conversely, the higher the intensity of its jumps is. So, there is a conflict between the approximation of the jump component increments and the complexity of its simulation procedure (when there are too many jumps). The choice of the truncation threshold is the result of a compromise between these constraints. It is time varying depending on the sequence (γ_n) and on the Lévy measure. When the jump component has integrable variation, we show that it is possible to find a compromise which preserves the best rate of the exact Euler scheme. We mean that it is possible to construct a step sequence (γ_n) and a sequence of truncation thresholds such that on the one hand, the best rate induced by this type of approximation is of order $n^{\frac{1}{3}}$ (see Proposition 2) and on the other hand the mean number of jumps at each time step remains uniformly bounded. This implies that the algorithm has a linear mean-complexity. Otherwise, this constraint of simulation slows down the best achievable rate. In particular, when the local behavior of the jump component is very irregular, Scheme (P) provides some very slow rates of convergence.

We propose to overcome this problem by adapting a work by Asmussen and Rosinski ([AsRo01]) in which it is shown that when the truncation threshold tends to 0, the small jump component of a one-dimensional Lévy process has asymptotically a Brownian behavior. It can be extended to d-dimensional Lévy processes (see Cohen and Rosinski, [CoRo05]). We then construct another Euler scheme (see Scheme (W)) by a *wienerization of the small jumps*. For this scheme, the compromise between the simulation and the approximation of the jump component is less constraining. Actually, we show that if the jump component has 3/2-integrable variation, it is possible to preserve the rate of $n^{\frac{1}{3}}$ and to respect the constraint of simulation. Furthermore, if π is symmetric in a neighborhood of 0, the preceding assertion is valid without any conditions on the small jumps (see Theorem 2 and Proposition 2).

Before outlining the structure of the paper, we list some notations:

• The set $\mathbb{M}_{d,l}$ of matrices with d rows and l columns and real-valued entries will be endowed with the norm $\|M\| := \sup_{\{|x|\leq 1\}} |Mx|/|x|$.

• For $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$, $x^{\otimes k}$ denotes the element of $(\mathbb{R}^d)^k$ defined by $x^{\otimes k}_{i_1,\ldots,i_k} = x_{i_1} x_{i_2} \ldots x_{i_n}$ for every $i_1, \ldots i_k \in \{1, \ldots, k\}$.

• For every $\mathcal{C}^k$-function $f : \mathbb{R}^d \mapsto \mathbb{R}$ and $x, y \in \mathbb{R}^d$, we adopt the following notation :

$$D^k f(x) y^{\otimes k} = \sum_{i_1,\ldots,i_k \in \{1,\ldots,k\}} \frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}(x) y_{i_1} \ldots y_{i_k}.$$

If $D^k f$ is bounded, we set

$$\|D^k f\|_\infty = \sup_{i_1,\ldots,i_k \in \{1,\ldots,k\}} \sup_{x \in \mathbb{R}^d} |\frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}(x)|.$$

• We say that $V : \mathbb{R}^d \mapsto \mathbb{R}_+^*$ is an EQ-function (for *Essentially Quadratic* function) if V is a $\mathcal{C}^2$-function such that $\lim V(x) = +\infty$ when $|x| \to +\infty$, $|\nabla V| \leq C\sqrt{V}$ and $D^2 V$ is bounded.
• We set $\Gamma_n = \sum_{k=1}^n \gamma_k$, and for $s > 0$, $\Gamma_n^{(s)} = \sum_{k=1}^n \gamma_k^{(s)}$.

In Section 2, we introduce the framework and the algorithm, and we recall a result of convergence of the sequence of empirical measures established in [Pan05]. In Section 3, we state our main results about the rate of convergence induced by the exact and approximate Euler schemes when the Lévy process has moments higher than 4. Sections 4 (resp. 5) are devoted to the proof of these results in the exact case (resp. approximate case). In Section 6, we state a partial extension of the main results when the Lévy process has less moments. Finally, in Section 7, we propose some numerical illustrations of our theoretical results.

2 Setting and Background on convergence results

For a Lévy measure π on $\mathbb{R}^l$, we denote by $(\mathbf{H_p})$ the following moment assumption

$$(\mathbf{H_p}) \quad : \quad \int_{|y|>1} |y|^{2p} \pi(dy) < +\infty \quad \text{with } p \geq 1.$$

We recall that a Lévy process with Lévy measure π is $2p$-integrable (see *e.g.* [BaMiRe01], Theorem 6.1). In [Pan05], we studied the convergence to the invariant measure for every $p > 0$. Here, we only consider the $p \geq 1$ case because our main problem is to observe the impact of the approximation of the jump component which only depends on the small jumps.
Throughout this paper, we denote by $(X_t)_{t\geq 0}$ a solution to the following SDE

$$dX_t = b(X_{t^-})dt + \sigma(X_{t^-})dW_t + \kappa(X_{t^-})dZ_t \tag{2}$$

where $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{M}_{d,l}$ and $\kappa : \mathbb{R}^d \mapsto \mathbb{M}_{d,l}$ are continuous with sublinear growth, $(W_t)_{t\geq 0}$ is a l-dimensional Brownian motion and $(Z_t)_{t\geq 0}$ is a locally square-integrable purely discontinuous $\mathbb{R}^l$-valued Lévy process independent of $(W_t)_{t\geq 0}$ with Lévy measure π and characteristic function given for every $t \geq 0$ by

$$\mathbb{E}\{e^{i<u,Z_t>}\} = \exp\left[t\left(\int e^{i<u,y>} - 1 - i < u, y > \pi(dy)\right)\right].$$

We recall that $(Z_t)_{t\geq 0}$ is a CCPP if and only if π is a finite measure and that, otherwise, it can be constructed as a limit of CCPP: let $(u_n)_{n\geq 1}$ be a sequence of positive numbers converging to 0. Let $D_n = \{|y| > u_n\}$ and let $((Z_{t,n})_{t\geq 0})_{n\geq 1}$ denote the sequence of processes defined by

$$Z_{t,n} := \sum_{0<s\leq t} \Delta Z_s 1_{\{\Delta Z_s \in D_n\}} - t\int_{D_n} y\pi(dy) \qquad \forall t\geq 0. \tag{3}$$

For every $n\geq 1$, $(Z_{t,n})_{t\geq 0}$ is a CCPP with intensity $\lambda_n = \pi(D_n)$ and jump size distribution $\mu_n(dx) = 1_{D_n}\frac{\pi(dx)}{\pi(D_n)}$. Furthermore, $Z_{.,n} \xrightarrow{n\to+\infty} Z$ in L^2 locally uniformly, *i.e.*

$$\mathbb{E}\{\sup_{0\leq t\leq T}|Z_t - Z_{t,n}|^2\} \xrightarrow{n\to+\infty} 0 \qquad \forall T>0.$$

Discretization of the SDE. We introduce three Euler schemes. Scheme (E) is constructed with the exact increments of the jump component and is called the exact Euler scheme. Schemes (P) and (W) are approximate Euler schemes. In scheme (P), we truncate the small jumps and in scheme (W), we refine the approximation by a *wienerization* of the small jumps.

Let $(\gamma_n)_{n\geq 1}$ be a decreasing sequence of positive numbers such that $\lim\gamma_n = 0$ and such that $\Gamma_n \to +\infty$. Let $(U_n)_{n\geq 1}$ be a sequence of i.i.d. square integrable centered $\mathbb{R}^l$-valued random variables such that $\Sigma_{U_1} = I_l$. Finally, let $(\bar{Z}_n)_{n\geq 1}$, $(\bar{Z}_n^P)_{n\geq 1}$ and $(\bar{Z}_n^W)_{n\geq 1}$ be sequences of independent $\mathbb{R}^l$-valued random variables, independent of $(U_n)_{n\geq 1}$ satisfying

$$\bar{Z}_n \stackrel{\mathcal{L}}{=} Z_{\gamma_n}, \qquad \bar{Z}_n^P \stackrel{\mathcal{L}}{=} Z_{\gamma_n,n} \quad \text{and} \quad \bar{Z}_n^W \stackrel{\mathcal{L}}{=} \bar{Z}_n^P + \sqrt{\gamma_n}Q_n\Lambda_n \qquad \forall n\geq 1,$$

where $(\Lambda_n)_{n\geq 1}$ is a sequence of i.i.d. random variables, independent of $(\bar{Z}_n^P)_{n\geq 1}, U_n)_{n\geq 1}$, such that $\mathbb{E}\Lambda_1 = 0$, $\Sigma_{\Lambda_1} = I_d$ and $\mathbb{E}\{\Lambda_1^{\otimes 3}\} = 0$, and (Q_n) is a sequence of $l\times l$ matrices such that

$$(Q_nQ_n^*)_{i,j} = \int_{|y|\leq u_k} y_iy_j\pi(dy).$$

We then denote by $(\bar{X}_n)$, $(\bar{X}_n^P)$ and $(\bar{X}_n^W)$, the Euler schemes recursively defined by $\bar{X}_0 = \bar{X}_0^P = \bar{X}_0^W = x\in\mathbb{R}^d$ and

$$\bar{X}_{n+1} = \bar{X}_n + \gamma_{n+1}b(\bar{X}_n) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n)U_{n+1} + \kappa(\bar{X}_n)\bar{Z}_{n+1} \tag{\textbf{E}}$$

$$\bar{X}_{n+1}^P = \bar{X}_n^P + \gamma_{n+1}b(\bar{X}_n^P) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n^P)U_{n+1} + \kappa(\bar{X}_n^P)\bar{Z}_{n+1}^P \tag{\textbf{P}}$$

$$\bar{X}_{n+1}^W = \bar{X}_n^W + \gamma_{n+1}b(\bar{X}_n^W) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_n^W)U_{n+1} + \kappa(\bar{X}_n^W)\bar{Z}_{n+1}^W. \tag{\textbf{W}}$$

We denote by $(\mathcal{F}_n)$, $(\mathcal{F}_n^P)$ and $(\mathcal{F}_n^W)$ the natural filtrations induced by $(\bar{X}_n)$, $(\bar{X}_n^P)$ and $(\bar{X}_n^W)$ respectively.

Remark 1. Note that $\bar{Z}_n^P$ can be simulated if both the intensity and the jump distribution of $(Z_{t,n})_{t\geq 0}$ can be computed. Its simulation time depends on the number of jumps of $(Z_{t,n})_t$ on $[0,\gamma_n]$. Its mean is $\pi(D_n)\gamma_n$. In order to ensure the linear mean-complexity of the algorithm, we ask in practice these means to be bounded, *i.e.*

$$\sup_{n\geq 1}\pi(D_n)\gamma_n < +\infty. \tag{4}$$

In scheme (W), Q_n can be computed by the Choleski method as an upper triangular matrix if $Q_nQ_n^*$ is definite. Otherwise, we can compute the principal square root of $Q_nQ_n^*$.

The associated sequences of empirical measures are defined by

$$\bar{\nu}_n = \frac{1}{H_n}\sum_{k=1}^n \eta_k\delta_{\bar{X}_{k-1}} \quad \bar{\nu}_n^P = \frac{1}{H_n}\sum_{k=1}^n \eta_k\delta_{\bar{X}_{k-1}^P} \quad \text{and} \quad \bar{\nu}_n^W = \frac{1}{H_n}\sum_{k=1}^n \eta_k\delta_{\bar{X}_{k-1}^W} \tag{5}$$

where (η_k) is a sequence of positive numbers such that $H_n = \sum_{k=1}^n \eta_k \xrightarrow{n\to+\infty} +\infty$.

Remark 2. As already mentioned, the rate of convergence will be only studied in the case $\eta_k = \gamma_k$. However, in the proof, we will intensively make use of convergence results for more general weighted empirical measures. That is why Proposition 1 is recalled in quite a general setting.

Let us pass now to the Lyapunov mean-reverting assumption. Let $a \in (0, 1]$ be a parameter relative to the mean-reversion intensity. Let $r \geq 0$ be a parameter relative to the growth of the noise coefficients σ and κ. In the sequel, we assume that there exists an EQ-function V such that

$$\textbf{Assumption } (\mathbf{S_{a,r}}): \qquad |b|^2 \leq CV^a \quad \mathrm{Tr}(\sigma\sigma^*) + \|\kappa\|^2 \leq CV^r \ \text{ with } r < a.$$
$$\textbf{Assumption } (\mathbf{R_a}): \qquad \langle \nabla V, b\rangle \leq \bar{\beta} - \bar{\alpha}V^a \quad \text{ with } \bar{\alpha} > 0 \text{ and } \bar{\beta} \in \mathbb{R}.$$

The first deals with the growth control of the coefficients and the second is called the mean-reverting assumption. These two assumptions imply assumptions $(\mathbf{S_{a,p,q}})$ and $(\mathbf{R_{a,p,q}})$ introduced in [Pan05]. Hence, we derive the following result from [Pan05]:

Proposition 1. *Let $a \in (0, 1]$, $p \geq 1$ and $r \in [0, a)$. Assume $(\mathbf{H_p})$, $(\mathbf{R_a})$ and $(\mathbf{S_{a,r}})$. Assume $\mathbb{E}\{|U_1|^{2p}\} + \mathbb{E}\{|\Lambda_1|^{2p}\} < +\infty$ and (η_n/γ_n) nonincreasing.*
(a) i. Then,

$$\sup_{n\geq 1} \bar{\nu}_n(V^{\frac{p}{2}+a-1}) < +\infty \quad a.s. \tag{6}$$

Hence, the sequence $(\bar{\nu}_n)_{n\geq 1}$ is a.s. tight as soon as $p/2 + a - 1 > 0$.
ii. Moreover, if $\kappa(x) \overset{|x|\to+\infty}{=} o(|x|)$ and $\mathrm{Tr}(\sigma\sigma^*) + \|\kappa\|^2 \leq CV^{\frac{p}{2}+a-1}$, then every weak limit of $(\bar{\nu}_n)$ is an invariant probability for the SDE (2). In particular, if $(X_t)_{t\geq 0}$ admits a unique invariant probability ν, then for every continuous function f such that $f = o(V^{\frac{p}{2}+a-1})$, $\lim_{n\to\infty} \bar{\nu}_n(f) = \nu(f)$.
iii. Furthermore, $\mathbb{E}\{V^p(\bar{X}_n)\} = O(\Gamma_n)$ and if $a = 1$, $\sup_{n\geq 1} \mathbb{E}\{V^p(\bar{X}_n)\} < +\infty$.
(b) The same result holds for $(\bar{\nu}^P_n)_{n\geq 1}$ and $(\bar{\nu}^W_n)_{n\geq 1}$.

Remark 3. For schemes (E) and (P), the above proposition is a direct consequence of Theorem 2 and Proposition 2 of [Pan05]. We did not study scheme (W) in [Pan05] but it is straightforward to show that the proposition holds true with a similar proof as that used for scheme (P). Note that when $a = 1$, $n \mapsto \mathbb{E}\{V^p(\bar{X}_n)\}$ is bounded whereas when $a < 1$, *i.e.* when the intensity of the mean-reverting is weak, one only has a control of its growth. This induces some technicalities but has no significant influence on the main results.

3 Main results

In this section, we suppose that $\mathbb{E}|Z_t|^{2p} < +\infty$ with $p > 2$ (see Section 6 for an extension to $p \in [1, 2)$). Let A denote the infinitesimal generator of (X_t). A is given for every $\mathcal{C}^2$-function f with bounded second derivatives[1] by,

$$\begin{aligned} Af(x) &= \langle \nabla f, b\rangle(x) + \frac{1}{2}Tr(\sigma^* D^2 f\sigma)(x) \\ &\quad + \int \big(f(x + \kappa(x)y) - f(x) - \langle \nabla f(x), \kappa(x)y\rangle 1_{\{|y|\leq 1\}}\big)\, \pi(dy). \end{aligned}$$

[1]Note that for such function, Af is well-defined since $\mathbb{E}|Z_t|^2 < +\infty$.

We evaluate the rate of convergence on some test functions g such that $g = Af + C$ where C is a nonnegative real number and f satisfies the following assumption:

$(\mathbf{C_f^p})$: (i) $f \in \mathcal{C}^4(\mathbb{R}^d)$ and $f(x) = O(V(x))$ as $|x| \to +\infty$.
(ii) For $k = 2, 3, 4$, $D^k f$ is a bounded and Lipschitz function.
(iii) $|\nabla f(x)|^2 = O(V^{\frac{\epsilon}{2}}(x))$ as $|x| \to +\infty$ with $\epsilon \in [0, p/2 + a - 1 - r)$.

Since ν is invariant for the SDE (2), we know that $\nu(Af) = 0$ (see *e.g.* [Pag01]) and then, $\nu(g) = \nu(Af + C) = C$. It follows that it suffices to evaluate the rate when $C = 0$.

Remark 4. For a jump diffusion like (2), we are not able to characterize simply the set of functions g which can be represented as $g = Af + C$ with f satisfying $(\mathbf{C_f^p})$. However, in the case of Brownian diffusions processes, some important works have been done in that direction. Actually, in [PaVe01], [PaVe03] and [PaVe06], Pardoux and Veretennikov show that in a Sobolev framework, existence and unicity hold for the Poisson equation $g - \nu(g) = Af$ where A is the infinitesimal generator of a positive recurrent diffusion. Moreover, in [LaPa02], Lamberton and Pagès show that when the diffusion is an Ornstein-Uhlenbeck process, the above equation can be solved in $\mathcal{C}^2(\mathbb{R}^d)$.

For this class of functions, the global structure of the rates of convergence is elucidated. For Scheme (E), our main result is Theorem 1. We show that for every sequence (γ_n), there exists a sequence (ρ_n) such that $(\rho_n \bar{\nu}_n^E(Af))$ converges weakly: a fast-decreasing sequence (γ_n) (in a sense being precised in Theorem 1(a)) leads to a CLT and a slowly-decreasing sequence (γ_n) leads to a convergence in probability to a deterministic constant (see Theorem 1(b)). The rate (ρ_n) is maximal for a "critical" choice of (γ_n) for which both types of convergence occur simultaneously. In particular, if $\gamma_n = \gamma_1 n^{-\zeta}$ with $\zeta \in (0, 1]$, the best rate holds for $\zeta = \frac{1}{3}$ (see "Particular Case"). In this case, ρ_n is of order $n^{\frac{1}{3}}$.
As concerns the approximate Euler schemes, our main results are Theorem 2 and Proposition 2. In the first one, we describe the structure of the rate induced by Scheme (P) and (W) as a function of (γ_n) and of (u_n). When (u_n) decreases "sufficiently fast" in a sense depending on the choice of the scheme, on (γ_n) and on the Lévy measure, the result induced by Scheme (E) remains valid for schemes (P) and (W). Otherwise, the approximation of the jump component dictates a slower rate of convergence.
Theorem 2 can not be directly applied in practice because it does not specify whether the fundamental condition of simulation (4) is compatible with the theoretical results. This is the purpose of Proposition 2 in which we give the best possible rates for schemes (P) and (W) under condition (4) as a function depending on the local behavior of the small jumps. In particular, Proposition 2 clarifies the impact of the wienerization of the small jumps announced in the introduction and shows that it makes possible to preserve the same rate of convergence of the exact Euler scheme for a wide class of Lévy processes (for which the exact simulation of the increments is impossible).

Let $f \in C^1(\mathbb{R}^d)$. We define $\tilde{H}^f$ by

$$\tilde{H}^f(z, x, y) = f(z + \kappa(x)y) - f(z) - \langle \nabla f(z), \kappa(x)y \rangle$$

and $z \mapsto \tilde{H}^f(z, x, y)$ is denoted by $\tilde{H}^f_{.,x,y}$. Our first result is the following:

Theorem 1. *Assume that* $\mathbb{E}|Z_t|^{2p} < +\infty$ *with* $p > 2$ *and that* (2) *admits a unique invariant measure* ν. *Let* $a \in (0, 1]$ *and* $r \geq 0$ *such that* $(\mathbf{R_a})$ *and* $(\mathbf{S_{a,r}})$ *are satisfied and*

$p/2+a-1>2r$. If moreover, $\mathbb{E}\{U_1^{\otimes 3}\}=0$, $\mathbb{E}\{|U_1|^{2p}\}<+\infty$ and $\eta_n=\gamma_n$ for every $n\geq 1$, then for every function $f:\mathbb{R}^d\mapsto\mathbb{R}$ satisfying $(\mathbf{C_f^p})$,

(a) If $\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}}\xrightarrow{n\to+\infty}\hat{\gamma}\in[0,+\infty)$, $\sqrt{\Gamma_n}\bar{\nu}_n(Af)\underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}}\mathcal{N}\Big(\hat{\gamma}m,\hat{\sigma}_f^2\Big)$.

(b) If $\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}}\xrightarrow{n\to+\infty}+\infty$, $\frac{\Gamma_n}{\Gamma_n^{(2)}}\bar{\nu}_n(Af)\underset{n\to+\infty}{\overset{\mathbb{P}}{\longrightarrow}}m$

where $\hat{\sigma}_f^2=\int\big(|\sigma^*\nabla f|^2(x)+\int\big(f(x+\kappa(x)y)-f(x)\big)^2\pi(dy)\big)\nu(dx)$ *and,*

$$m=-\int(\phi_1(x)+\phi_2(x)+\phi_3(x))\nu(dx)\text{ with }\phi_1(x)=\frac{1}{2}D^2f(x)b(x)^{\otimes 2},$$

$$\phi_2(x)=\int\frac{1}{6}D^3f(x);b(x);(\sigma(x)u)^{\otimes 2}+\frac{1}{24}D^4f(x)(\sigma(x)u)^{\otimes 4}\mathbb{P}_{U_1}(du)$$

$$\textit{and,}\quad\phi_3(x)=\frac{1}{2}\int\pi(dy_1)\int\pi(dy_2)\tilde{H}^{\tilde{H}^f_{.,x,y_1}}(x,x,y_2)$$

$$+\int\pi(dy_1)\Big(\langle\nabla\tilde{H}^f_{.,x,y_1}(x),b(x)\rangle+\int\mathbb{P}_{U_1}(du)D^2(\tilde{H}^f_{.,x,y_1})(x)(\sigma(x)u)^{\otimes 2}\Big).$$

Particular Case. Assume that $\gamma_n=\gamma_1 n^{-\zeta}$ with $\zeta\in(0,1]$. Then,

$$\begin{cases}\sqrt{\gamma_1\log n}\,\bar{\nu}_n(Af)\underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}}\mathcal{N}\Big(0,\hat{\sigma}_f^2\Big). & \text{if }\zeta=1\\ \sqrt{\frac{\gamma_1}{1-\zeta}}n^{\frac{1-\zeta}{2}}\bar{\nu}_n(Af)\underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}}\mathcal{N}\Big(\hat{\gamma}m,\hat{\sigma}_f^2\Big). & \text{if }\zeta\in[1/3,1)\\ \frac{1-2\zeta}{\gamma_1(1-\zeta)}n^{\zeta}\bar{\nu}_n(Af)\underset{n\to+\infty}{\overset{\mathbb{P}}{\longrightarrow}}m & \text{if }\zeta<1/3\end{cases}$$

where $\hat{\gamma}=0$ if $\zeta\in(1/3,1)$ and $\hat{\gamma}=\sqrt{6\gamma_1^3}$ if $\zeta=1/3$. On Figure 1, one represents $\zeta\mapsto h(\zeta)$ where $h(\zeta)$ denotes the exponent of the rate. One observes that $\max_{\zeta\in(0,1]}h(\zeta)=h(1/3)=1/3$.

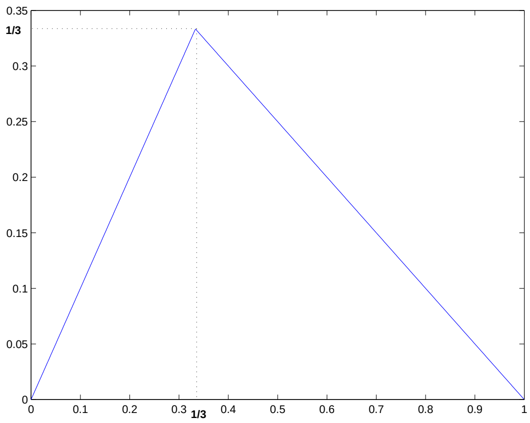

Figure 1: Rate of convergence for polynomial steps

Remark 5. Theorem 1 shows that the rate is the same as that obtained for Brownian diffusions. In particular, when $\kappa=0$, Theorem 1 extends the rate results of [LaPa02] and [Lem06] to the weakly mean-reverting diffusions ($a<1$), whose convergence to the invariant measure has been studied in [LaPa03].

Note that the condition $\mathbb{E}\{U_1^{\otimes 3}\}=0$ is not necessary for the convergence of the empirical measures but plays a role in the rate. Without this condition, the best rate would be of order $n^{\frac{1}{4}}$, obtained for $\zeta=1/2$ (see [LaPa02] in the case of Brownian diffusions).

Let us pass now to the main results for the approximate Euler schemes. Let $(u_k)_{k\geq 1}$ denote the sequence of truncation thresholds and set $\beta^{(s)}_{n,\pi} = \sum_{k=1}^{n}\gamma_k\int_{|y|\leq u_k}|y|^s\pi(dy)$. For $s\in\{2,3,4\}$, we introduce a new assumption $(\mathbf{A_s^1})$ which is relative to the impact of the jump component approximation as a function of the steps and of the truncation thresholds:

$$(\mathbf{A_s^1}): \quad \frac{\beta^{(s)}_{n,\pi}}{\Gamma^{(2)}_n} \xrightarrow{n\to+\infty} \hat{\alpha}_s\in[0,+\infty] \quad \text{and} \quad \frac{\beta^{(s)}_{n,\pi}}{\sqrt{\Gamma_n}} \xrightarrow{n\to+\infty} \hat{\beta}_s\in[0,+\infty].$$

Since $s\mapsto\beta^{(s)}_{n,\pi}$ is a decreasing function, $s\mapsto\hat{\alpha}_s$ and $s\mapsto\hat{\beta}_s$ both decrease. This can be interpreted as follows: the constraint on (u_k) decreases with s.
For $s\in\{2,3,4\}$, we also introduce another assumption on the Lévy measure that will be necessary to transform some tightness results in some convergence in distribution results:

$$(\mathbf{A_s^2}): \text{ For every } i_1,\ldots,i_s\in\{1,\ldots,l\},\ \Big(\frac{\int_{|y|\leq u_k} y_{i_1}\ldots y_{i_s}\pi(dy)}{\int_{|y|\leq u_k}|y|^s\pi(dy)}\Big)_{k\geq 1} \text{ converges in } \mathbb{R}.$$

For instance, the above assumption is satisfied if $\pi(dy)=\psi(y)\lambda_l(dy)$ where ψ satisfies: there exists $\alpha\in[l,l+2)$ such that $|y|^{\alpha+l}\psi(y)\to C_0\in\mathbb{R}_+^*$ when $y\to 0$.
Throughout this paper, we will say that the Lévy measure π *is quasi-symmetric in a neighborhood of 0* if $\int_{\{|y|\leq u\}} y^{\otimes 3}\pi(dy)=0$ for u sufficiently small. In particular, this assertion holds if π is symmetric in a neighborhood of 0.
We will also say that a real-valued random variable X is *quasi-subgaussian* if there exists $m>0$ and $\sigma>0$ such that for all $M>0$

$$\mathbb{P}(|X|>M)\leq\mathbb{P}(|Y|+m>M) \quad \text{with} \quad Y\sim\mathcal{N}(0,\sigma^2).$$

Theorem 2. *Let $a\in(0,1]$, $r\geq 0$, $p<2$ such that the conditions of Theorem 1 are satisfied. Assume that $\mathbb{E}\{|\Lambda_1|^{2p}\}<+\infty$ and that $(u_k)_{k\geq 1}$ is decreases to 0.*
(a) i. Scheme (P): Assume that $(\mathbf{A_s^1})$ holds with $s=2$.
- *If $\hat{\alpha}_s=0$ or $\hat{\beta}_s=0$, then the conclusions of Theorem 1 are still valid for $(\bar{\nu}_n^P)$.*
- *If $\hat{\alpha}_s\in(0,+\infty]$ and $\hat{\beta}_s\in(0,+\infty)$, then $(\frac{\Gamma_n}{\beta^{(s)}_{n,\pi}}\bar{\nu}_n^P(Af))_{n\geq 1}$ is tight with quasi-subgaussian limiting distributions.*
- *If $\hat{\alpha}_s\in(0,+\infty]$ and $\hat{\beta}_s=+\infty$, then $(\frac{\Gamma_n}{\beta^{(s)}_{n,\pi}}\bar{\nu}_n^P(Af))_{n\geq 1}$ is tight with bounded limiting distributions.*

ii. Scheme (W): Assume that $(\mathbf{A_3^1})$ holds.
Then, the conclusions of $(a).i$ are valid for $(\bar{\nu}_n^W(Af))_{n\geq 1}$ with $s=3$. Furthermore, if π is quasi-symmetric in a neighborhood of 0 and $(\mathbf{A_4^1})$ holds, the conclusions of $(a).i$ are valid for $(\bar{\nu}_n^W(Af))_{n\geq 1}$ with $s=4$.

(b) i. Scheme (P): Assume that $(\mathbf{A_s^1})$ and $(\mathbf{A_s^2})$ hold with $s=2$. Then,

$$\frac{\Gamma_n}{\beta^{(s)}_{n,\pi}}\bar{\nu}_n^P(Af) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}\Big(m/\hat{\alpha}_s-m_s,(\hat{\sigma}_f/\hat{\beta}_s)^2\Big) \qquad \text{if } \hat{\alpha}_s\in(0,+\infty] \text{ and } \hat{\beta}_s\in(0,+\infty)$$

$$\frac{\Gamma_n}{\beta^{(s)}_{n,\pi}}\bar{\nu}_n^P(Af) \xrightarrow[n\to+\infty]{\mathbb{P}} m/\hat{\alpha}_s-m_s \qquad \text{if } \hat{\alpha}_s\in(0,+\infty] \text{ and } \hat{\beta}_s\in(0,+\infty),$$

with $|m_2|\leq\bar{m}_2=(d/2)\|D^2f\|_\infty\int\|\kappa\|^2(x)\nu(dx)$ and m and $\hat{\sigma}_f^2$ like in Theorem 1.
ii. Scheme (W): Assume that $(\mathbf{A_3^1})$ and $(\mathbf{A_3^2})$ hold.
Then, the conclusions of $(b).i$ are valid for $(\bar{\nu}_n^W(Af))_{n\geq 1}$ with $s=3$ and a real number m_3

satisfying $|m_3| \leq \bar{m}_3 = (d^{\frac{3}{2}}/6)\|D^3 f\|_\infty \int \|\kappa(x)\|^2 \nu(dx)$.
Furthermore, if π *is quasi-symmetric in a neighborhood of 0 and if* $(\mathbf{A_4^1})$ *and* $(\mathbf{A_4^2})$ *hold, the conclusions of* $(b).i$ *are valid for* $(\bar{\nu}_n^W(Af))_{n\geq 1}$ *with* $s = 4$ *and a real number* m_4 *satisfying* $|m_4| \leq \bar{m}_4 = (d^2/24)\|D^4 f\|_\infty \int \|\kappa(x)\|^4 \nu(dx)$.

Remark 6. Note that in the one-dimensional case, Assumption $(\mathbf{A_s^2})$ is always satisfied when $s = 2$ or $s = 4$. In those cases, $m_s = \frac{1}{s!}\int f^{(s)}(x)\kappa(x)^s \nu(dx)$. If $s = 3$, Assumption $(\mathbf{A_s^2})$ is satisfied if $\int_{\{|y|\leq u_k} y^3 \pi(dy)/\int_{\{|y|\leq u_k} |y|^3 \pi(dy) \to a_3 \in \mathbb{R}$. In this case, $m_3 = a_3 \frac{1}{3!}\int f^{(3)}(x)\kappa(x)^3 \nu(dx)$. In the multidimensional case, the value of m_s is also explicit but its expression is more complicated (see proof of Lemma 7).

Let us now state Proposition 2. In (a), we provide some conditions on the Lévy measure in the neighborhood of 0 which preserve the rate of convergence induced by the exact Euler scheme under the condition of simulation (4). In (b), we suppose that the Lévy measure has a density closed to that of an α-stable process in the neighborhood of 0 and give in that case the optimal rate for the two schemes as a function of α. For these two parts, we also give some available choices of steps and truncation thresholds.

Proposition 2. *Let* $a \in (0, 1]$, $r \geq 0$, $p \geq 2$ *such that the conditions of Theorem 1 are satisfied. Assume that* $\mathbb{E}\{|\Lambda_1|^{2p}\} < +\infty$.
(a) Assume that $\int_{\{|y|\leq 1\}} |y|^q \pi(dy) < +\infty$ *with* $q \in [0, 2]$ *and set* $\gamma_k = \gamma_1 k^{-\frac{1}{3}}$ *and* $u_k = \gamma_k^r$ *with* $r \in [\frac{1}{q}, \frac{1}{s-q}]$. *Then, Condition* (4) *holds and,*
i. Scheme (P): If $q \leq 1$ *and* $s = 2$, $n^{\frac{1}{3}}\bar{\nu}_n^P(Af) \underset{n\to+\infty}{\overset{\mathcal{L}}{\to}} \sqrt{2/3}\mathcal{N}\big(m\sqrt{6}, \hat{\sigma}_f^2\big)$.

ii. Scheme (W): If $q \leq 3/2$ *and* $s = 3$, $n^{\frac{1}{3}}\bar{\nu}_n^W(Af) \underset{n\to+\infty}{\overset{\mathcal{L}}{\to}} \sqrt{2/3}\mathcal{N}\big(m\sqrt{6}, \hat{\sigma}_f^2\big)$.
Furthermore, if π *is quasi-symmetric in the neighborhood of 0, the preceding assertion is valid with* $s = 4$ *and every* $q \in [0, 2]$.
(b) Assume that there exists $\epsilon_0 > 0$ *such that*

$$\pi(dy) = \psi(y)\lambda_l(dy) \quad \text{with} \quad 1_{\{0<|y|\leq\epsilon_0\}}\frac{C_1}{|y|^{\alpha+l}} \leq \psi(y) \leq \frac{C_2}{|y|^{\alpha+l}}1_{\{0<|y|\leq\epsilon_0\}}. \tag{7}$$

Set $\gamma_k = \gamma_1 k^{-(\frac{1}{3}\vee\frac{\alpha}{2s-\alpha})}$, $u_k = \gamma_k^r$ *with* $r \in [\frac{1}{\alpha}, \frac{1}{(s-\alpha)\vee\alpha}]$. *Then, Condition* (4) *holds and,*
i. Scheme (P), $s = 2$: $\big(n^{(\frac{1}{3}\wedge\frac{2-\alpha}{4-\alpha})}\bar{\nu}_n^P(Af)\big)_{n\geq 1}$ *is tight.*
ii. Scheme (W), $s = 3$: $\big(n^{(\frac{1}{3}\wedge\frac{3-\alpha}{6-\alpha})}\bar{\nu}_n^W(Af)\big)_{n\geq 1}$ *is tight.*

Remark 7. Figure 7 represents $\alpha \mapsto h(\alpha)$ where $h(\alpha)$ denotes the exponent of the optimal rate induced by each approximate scheme under the assumptions of Proposition 2(b). This figure emphasizes the necessity of scheme (W) when the jump component has infinite variation because the optimal rate of convergence induced by scheme (P) decreases very rapidly in that case.

Remark 8. The fact that we optimize the rate for the range of sequences $(\gamma_k, u_k)_{k\geq 1}$ such that the linearity of the mean-complexity of the procedure is ensured can be disputable when the optimal rate is not of order $n^{\frac{1}{3}}$. Actually, in this case, even if for a smaller level of truncation, the linearity of the complexity fails, the theoretical rate is better. Hence, another point of view consists in evaluating the order of precision as a function of the complexity. Some precise statements on that question would require some Berry-Esseen type estimates (in our inhomogeneous framework). Nevertheless, heuristic study can be done when (7) is satisfied and suggests that the asymptotic order of precision as a function of the mean-complexity is optimized for a class of steps and truncation levels including the choices of Proposition 2.

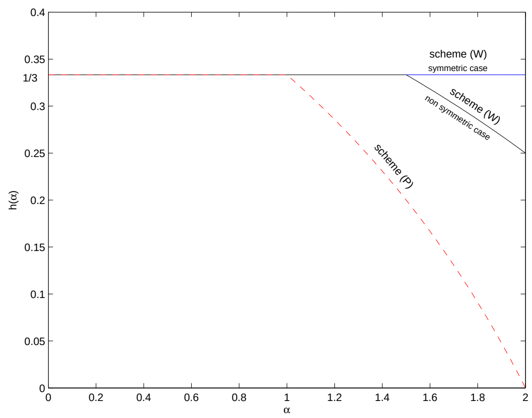

Figure 2: Optimal rate in terms of the local behavior of the Lévy process

4 Proof of Theorem 1

In this section, we prove the main result induced by the exact Euler scheme: Theorem 1. Firstly, we decompose $\bar{\nu}_n(Af)$ (see Lemma 1) and then, we compute the rate of each term of the decomposition in Lemmas 2, 4 and 3. (We will principally focus on Lemmas 2 and 3 where the rate of the jump part of the decomposition is studied). Finally, a synthesis of the previous lemmas is realized in subsection 4.2 and completes the proof of Theorem 1.

4.1 Decomposed computation of the rate of $\bar{\nu}_n(Af)$.

We set

$$\bar{X}_{k,1} = \bar{X}_{k-1} + \gamma_k b(\bar{X}_{k-1}), \quad \text{and} \quad \bar{X}_{k,2} = \bar{X}_{k,1} + \sqrt{\gamma_k}\sigma(\bar{X}_{k-1})U_k.$$

Denote by $(Z^{(k)})_{k\geq 1}$, a sequence of i.i.d. random variables such that $Z^{(1)} \overset{\mathcal{L}}{=} Z$ and set $\bar{Z}_k = Z^{(k)}_{\gamma_k}$.

Lemma 1. *For $f \in C^2(\mathbb{R}^d)$, we have the following decomposition.*

$$\sum_{k=1}^{n} \gamma_k Af(\bar{X}_{k-1}) = f(\bar{X}_n) - f(\bar{X}_0) - \sum_{k=1}^{n} \Big(\xi_1(\gamma_k, \bar{X}_{k-1}, U_k) + \xi_2(\gamma_k, \bar{X}_{k-1}, Z^{(k)})\Big)$$
$$- \sum_{k=1}^{n} \Big(\Theta_1(\gamma_k, \bar{X}_{k-1}) + \Theta_2(\gamma_k, \bar{X}_{k-1}, U_k) + \Theta_3(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)})\Big)$$
$$- \sum_{k=1}^{n} \Big((R_1 + R_2)(\gamma_k, \bar{X}_{k-1}, U_k) + R_3(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)})\Big)$$

where,

$$\xi_1(\gamma, x, U_k) = \sqrt{\gamma}\langle \nabla f(x), \sigma(x)U_k\rangle,$$
$$\xi_2(\gamma, x, Z) = \int_0^{\gamma} \langle \nabla f(x), \kappa(x)dZ_s\rangle + \big(\sum_{0<s\leq\gamma} \tilde{H}^f(x, x, \Delta Z_s) - \gamma \int \tilde{H}^f(x, x, y)\pi(dy)\big),$$
$$\Theta_1(\gamma, x) = \gamma \int_0^1 \langle \nabla f(x + \theta\gamma b(x)) - \nabla f(x), b(x)\rangle d\theta,$$
$$\Theta_2(\gamma, x, u) = \gamma \int_0^1 (1-\theta)\big(D^2 f(x + \gamma b(x) + \theta\sqrt{\gamma}\sigma(x)u) - D^2 f(x)\big)(\sigma(x)u)^{\otimes 2} d\theta,$$

$$
\begin{aligned}
\Theta_3(\gamma,x,z,Z) &= \sum_{0<s\leq\gamma}\big(\tilde{H}^f(x,z,Z_{s^-},\Delta Z_s)-\tilde{H}^f(x,x,0,\Delta Z_s)\big),\\
R_1(\gamma,x,U_k) &= \sqrt{\gamma}\langle\nabla f(x+\gamma b(x))-\nabla f(x),\sigma(x)U_k\rangle,\\
R_2(\gamma,x,U_k) &= \frac{\gamma}{2}\big(D^2f(x)(\sigma(x)U_k)^{\otimes 2}-\mathbb{E}\{D^2f(x)(\sigma(x)U_k)^{\otimes 2}\}\big),\\
R_3(\gamma,x,z,Z) &= \int_0^\gamma\langle\nabla f(z+\kappa(x)Z_{s^-})-\nabla f(x),\kappa(x)dZ_s\rangle.
\end{aligned}
$$

Proof. We write

$$
f(\bar{X}_k)-f(\bar{X}_{k-1})=\Big(f(\bar{X}_{k,1})-f(\bar{X}_{k-1})\Big)+\Big(f(\bar{X}_k^2)-f(\bar{X}_k^1)\Big)+\Big(f(\bar{X}_k)-f(\bar{X}_{k,2})\Big)
$$

We expand the first two terms by the Taylor formula and use the Itô formula (with jumps) for the last one. The lemma follows by summing up the equality for $k=1,\ldots,n$. $\square$

As mentioned before, we study successively the rate of convergence of each term of the previous decomposition. We start by showing a CLT for the terms associated with ξ_1 and ξ_2.

Lemma 2. *Assume that* $(\mathbf{H_p})$ *holds for* $p>2$. *Let* $f:\mathbb{R}^d\mapsto\mathbb{R}$ *satisfy* $(\mathbf{C_f^p})$. *Then, with the notations of Lemma 1, we have*
(a)

$$
\mathbb{E}\{|\xi_2(\gamma,x,Z)|^2\}=\gamma\int\big(f(x+\kappa(x)y)-f(x)\big)^2\pi(dy) \tag{8}
$$

and there exists $\delta>0$ *and a locally bounded function* C *such that*

$$
\mathbb{E}\{|\xi_2(\gamma,x,Z)|^{2(1+\delta)}\}\leq C(x)\gamma \tag{9}
$$

(b) Moreover, if $(\mathbf{R_a})$ *and* $(\mathbf{S_{a,r}})$ *hold with* $2r<p/2+a-1$ *and* $\mathbb{E}\{|U_1|^{2p}\}<+\infty$*, then,*

$$
\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n\Big(\xi_1(\gamma_k,\bar{X}_{k-1},U_k)+\xi_2(\gamma_k,\bar{X}_{k-1},Z^{(k)})\Big)\xrightarrow[n\to+\infty]{\mathcal{L}}\mathcal{N}\big(0,\hat{\sigma}_f^2\big),
$$

with $\hat{\sigma}_f^2=\int\Big(|\sigma^*\nabla f|^2(x)+\int\big(f(x+\kappa(x)y)-f(x)\big)^2\pi(dy)\Big)\nu(dx)$.

Proof. (a). Let $(Z_{.,n})_{n\geq 1}$ be the sequence of processes defined by (3). We know that

$$
\mathbb{E}\{\sup_{\{0\leq s\leq t\}}|Z_{s,n}-Z_s|^2\}\xrightarrow{n\to+\infty}0.
$$

As $Z_{.,n}$ has bounded variations, $\xi_2(\gamma,x,Z_{.,n})$ can be written

$$
\xi_2(\gamma,x,Z_{.,n})=\sum_{0<s\leq\gamma}1_{\{|\Delta Z_s|>u_n\}}\big(f(x+\kappa(x)\Delta Z_s)-f(x)\big)-\gamma\int_{\{|y|>u_n\}}\big(f(x+\kappa(x)y)-f(x)\big)\pi(dy).
$$

Since D^2f is bounded and $\mathbb{E}|Z_t|^4<+\infty$, one easily checks that $\xi_2(\gamma,x,Z_{.,n})$ is a locally square-integrable purely discontinuous martingale. We deduce from the compensation formula that

$$
\mathbb{E}\{|\xi_2(\gamma,x,Z_{.,n})|^2\}=\gamma\int_{\{|y|>u_n\}}|f\big(x+\kappa(x)y)\big)-f(x)|^2\pi(dy). \tag{10}
$$

We also check that

$$\mathbb{E}\{|\xi_2(\gamma,x,Z)-\xi_2(\gamma,x,Z_{.,n})|^2\}\le C_x\int_{\{|y|\le u_n\}}|y|^2\pi(dy)\xrightarrow{n\to+\infty}0.$$

Letting $n\to+\infty$ in (10) yields the first identity.
Now, let us prove the inequality. $\xi_2(\gamma,x,Z)=\langle\nabla f(x),\kappa(x)Z_\gamma\rangle+M_\gamma$ where M is a martingale defined by

$$\begin{aligned}M_\gamma=&\sum_{0<s\le\gamma}\int_0^1\langle\nabla f(x+\theta\kappa(x)\Delta Z_s)-\nabla f(x),\kappa(x)\Delta Z_s\rangle d\theta\\&-\gamma\int\int_0^1\langle\nabla f(x+\theta\kappa(x)y)-\nabla f(x),\kappa(x)y\rangle d\theta\pi(dy).\end{aligned}$$

Let $\delta\in(0,1]$ such that $4(1+\delta)\le 2p$. Since ∇f is Lipschitz continuous, we derive from the Burkholder-Davis-Gundy inequality that

$$\mathbb{E}|Z_\gamma|^{2(1+\delta)}\le\mathbb{E}\Big\{\big(\sum_{0<s\le\gamma}|\Delta Z_s|^2\big)^{1+\delta}\Big\}\quad\text{and}\quad\mathbb{E}|M_\gamma|^{2(1+\delta)}\le C(x)\mathbb{E}\Big\{\big(\sum_{0<s\le\gamma}|\Delta Z_s|^4\big)^{1+\delta}\Big\}.$$

It follows that

$$\mathbb{E}\{|\xi_2(\gamma,x,Z)|^{2(1+\delta)}\}\le C_1(x)\mathbb{E}\Big\{\big(\sum_{0<s\le\gamma}|\Delta Z_s|^2\big)^{1+\delta}\Big\}+C_2(x)\mathbb{E}\Big\{\big(\sum_{0<s\le\gamma}|\Delta Z_s|^4\big)^{1+\delta}\Big\}.$$

Then, it suffices to prove that

$$\mathbb{E}\Big\{\big(\sum_{0<s\le\gamma}|\Delta Z_s|^{2+\rho}\big)^{1+\delta}\Big\}=O(\gamma)\qquad\text{for }\rho=0\text{ and }\rho=2.\tag{11}$$

Denote by $(\tilde{M}_s)$ the martingale defined by $\tilde{M}_s=\sum_{0<s\le\gamma}|\Delta Z_s|^{2+\rho}-\gamma\int|y|^{2+\rho}\pi(dy)$. By the elementary inequality

$$\forall u,v\in\mathbb{R}\text{ and }\alpha>0,\quad|u+v|^\alpha\le 2^{\alpha\vee1-1}(|u|^\alpha+|v|^\alpha).\tag{12}$$

we have,

$$\mathbb{E}\Big\{\big(\sum_{0<s\le\gamma}|\Delta Z_s|^{2+\rho}\big)^{1+\delta}\Big\}\le C\big(\mathbb{E}|\tilde{M}_\gamma|^{1+\delta}+\gamma^{1+\delta}\int|y|^{2+\rho}\pi(dy)\big).$$

By the Burkhölder-Davis-Gundy inequality,

$$\mathbb{E}|\tilde{M}_\gamma|^{1+\delta}\le C\mathbb{E}\Big\{\big(\sum_{0<s\le\gamma}|\Delta Z_s|^{2(2+\rho)}\big)^{\frac{1+\delta}{2}}\Big\}.$$

Since $(1+\delta)/2\le1$, it follows from (12) and from the compensation formula that

$$\mathbb{E}|\tilde{M}_\gamma|^{1+\delta}\le C\mathbb{E}\Big\{\sum_{0<s\le\gamma}|\Delta Z_s|^{(2+\rho)(1+\delta)}\Big\}\le C\gamma\int|y|^{(2+\rho)(1+\delta)}\pi(dy).$$

Since $2\le(2+\rho)(1+\delta)\le 2p$, $\int|y|^{(2+\rho)(1+\delta)}\pi(dy)<+\infty$. (11) follows.

(b) Let $\{(\xi_k^n), k = 1, \dots, n, n \geq 1\}$ be a sequence of triangular arrays of square-integrable martingale increments defined by

$$\xi_k^n = \frac{1}{\sqrt{\Gamma_n}}\Big(\xi_1(\gamma_k, \bar{X}_{k-1}, U_k) + \xi_2(\gamma_k, \bar{X}_{k-1}, Z^{(k)})\Big).$$

Since $\Sigma_{U_1} = I_l$, we have

$$\mathbb{E}\{|\xi_1(\gamma_k, \bar{X}_{k-1}, U_k)|^2/\mathcal{F}_{k-1}\} = \gamma_k |\sigma^*\nabla f|^2(\bar{X}_{k-1}).$$

Moreover, $\xi_1(\gamma_k, \bar{X}_{k-1}, U_k)$ and $\xi_2(\gamma_k, \bar{X}_{k-1}, Z^{(k)})$ are independent conditionally to $\mathcal{F}_{k-1}$ and

$$\mathbb{E}\{\xi_1(\gamma_k, \bar{X}_{k-1}, U_k)/\mathcal{F}_{k-1}\} = \mathbb{E}\{\xi_2(\gamma_k, \bar{X}_{k-1}, Z^{(k)})/\mathcal{F}_{k-1}\} = 0.$$

Then, we deduce from (8) that

$$\begin{aligned}
\mathbb{E}\{|\xi_k^n|^2/\mathcal{F}_{k-1}\} &= \frac{1}{\Gamma_n}\Big(\mathbb{E}\{|\xi_1(\gamma_k, \bar{X}_{k-1}, U_k)|^2/\mathcal{F}_{k-1}\} + \mathbb{E}\{|\xi_2(\gamma_k, \bar{X}_{k-1}, Z^{(k)})|^2/\mathcal{F}_{k-1}\}\Big)\\
&= \frac{\gamma_k}{\Gamma_n}\Big(|\sigma^*\nabla f|^2(\bar{X}_{k-1}) + \int \big(f(\bar{X}_{k-1} + \kappa(\bar{X}_{k-1})y) - f(\bar{X}_{k-1})\big)^2\pi(dy)\Big).
\end{aligned}$$

Since $D^2 f$ is bounded, we derive from Taylor's formula and from the assumptions on r and on ∇f that

$$\int \big(f(. + \kappa(.)y) - f(.)\big)^2\pi(dy) + |\sigma^*\nabla f|^2 \leq CV^{(\epsilon+r)\vee(2r)} = o(V^{\frac{p}{2}+a-1}). \tag{13}$$

Hence, Proposition 1 yields

$$\sum_{k=1}^{n} \mathbb{E}\{|\xi_k^n|^2/\mathcal{F}_{k-1}\} \xrightarrow{n\to+\infty} \int \Big(|\sigma^*\nabla f|^2 + \int \big(f(. + \kappa(.)y) - f(.)\big)^2\pi(dy)\Big)d\nu. \tag{14}$$

Then, the lemma will follow from the central limit theorem for arrays of square-integrable martingale increments (see Hall and Heyde, [HaHe80]) provided the *Lindeberg condition* is fulfilled, *i.e.*

$$R_n^\rho = \sum_{k=1}^{n} \mathbb{E}\{|\xi_k^n|^2 1_{\{|\xi_k^n| \geq \rho\}}/\mathcal{F}_{k-1}\} \xrightarrow{n\to+\infty} 0 \qquad a.s. \quad \forall \rho > 0.$$

Let $A \in (0, +\infty)$ and set

$$\begin{aligned}
R_{n,1}^{\rho,A} &= \sum_{k=1}^{n} 1_{\{|\bar{X}_{k-1}| \leq A\}} \mathbb{E}\{|\xi_k^n|^2 1_{\{|\xi_k^n| \geq \rho\}}/\mathcal{F}_{k-1}\},\\
R_{n,2}^{\rho,A} &= \sum_{k=1}^{n} 1_{\{|\bar{X}_{k-1}| \geq A\}} \mathbb{E}\{|\xi_k^n|^2 1_{\{|\xi_k^n| \geq \rho\}}/\mathcal{F}_{k-1}\}.
\end{aligned}$$

We have $\mathbb{E}\{|\xi_k^n|^2 1_{\{|\xi_k^n| \geq \rho\}}/\mathcal{F}_{k-1}\} = F_A^n(\bar{X}_{k-1}, \gamma_k)$ where

$$F_A^n(x, \gamma) = \frac{1}{\Gamma_n}\mathbb{E}\{|\xi_1(\gamma, x, U_1) + \xi_2(\gamma, x, Z)|^2 1_{\{|\xi_1(\gamma,x,U_1) + \xi_2(\gamma,x,Z)| \geq \rho\sqrt{\Gamma_n}\}}\}.$$

Let $\delta > 0$ such that (9) holds. By setting $\bar{p} = 1 + \delta$ and $\bar{q} = \frac{1+\delta}{\delta}$, we derive from the Holder inequality that

$$F_A^n(x, \gamma) \leq \frac{1}{\Gamma_n}\mathbb{E}\{|\xi_1(\gamma, x, U_1) + \xi_2(\gamma, x, Z)|^{2(1+\delta)}\}^{\frac{1}{1+\delta}}\Big(\mathbb{P}(|\xi_1(\gamma, x, U_1) + \xi_2(\gamma, x, Z)| \geq \rho\sqrt{\Gamma_n})\Big)^{\frac{\delta}{1+\delta}}$$

On the one hand, we deduce from (12) and from (9) that

$$\mathbb{E}\{|\xi_1(\gamma,x,U_1)+\xi_2(\gamma,x,Z)|^{2(1+\delta)}\}^{\frac{1}{1+\delta}} \le C(x,\delta)(\gamma^{1+\delta}+\gamma)^{\frac{1}{1+\delta}} \le C_1(x,\delta)\gamma^{\frac{1}{1+\delta}}$$

where $x\mapsto C_1(x,\delta)$ is locally bounded. On the other hand, we deduce from the Chebyschev inequality that,

$$\begin{aligned}\Big(\mathbb{P}(|\xi_1(\gamma,x,U_1)+\xi_2(\gamma,x,Z)|\ge\rho\sqrt{\Gamma_n})\Big)^{\frac{\delta}{2+\delta}} &\le \frac{1}{(\rho^2\Gamma_n)^{\frac{\delta}{1+\delta}}}\mathbb{E}\{|\xi_1(\gamma,x,U_1)+\xi_2(\gamma,x,Z)|^2\}^{\frac{\delta}{1+\delta}}\\ &\le C_2(x,\delta,\rho)(\frac{\gamma}{\Gamma_n})^{\frac{\delta}{1+\delta}}\end{aligned}$$

where $x\mapsto C_2(x,\delta,\rho)$ is locally bounded. Then, for every $A>0$ and $\rho>0$,

$$R_{n,1}^{\rho,A}\le C_{A,\rho}\frac{1}{\Gamma_n^{1+\frac{\delta}{1+\delta}}}\sum_{k=1}^n\gamma_k = C_{A,\rho}\frac{1}{\Gamma_n^{\frac{\delta}{1+\delta}}}\xrightarrow{n\to+\infty}0 \qquad a.s.$$

Now, we observe $R_{n,2}^{\rho,A}$. From (13), we have : $\mathbb{E}\{|\xi_k^n|^2/\mathcal{F}_{k-1}\}\le CV^{\beta}(\bar{X}_{k-1})$ with $\beta< p/2+a-1$. Therefore,

$$R_{n,2}^{A,\rho}\le\sum_{k=1}^n 1_{\{|\bar{X}_{k-1}|\ge A\}}\mathbb{E}\{|\xi_k^n|^2/\mathcal{F}_{k-1}\}\le\sup_{|x|\ge A}\frac{V^{\beta}(x)}{V^{\frac{p}{2}+a-1}(x)}\sup_{n\in\mathbb{N}}\bar{\nu}_n(V^{\frac{p}{2}+a-1})=\phi(A)\bar{\nu}_n(V^{\frac{p}{2}+a-1})$$

where $\phi(A)\xrightarrow{A\to+\infty}0$. Since $\sup_{n\in\mathbb{N}}\bar{\nu}_n(V^{\frac{p}{2}+a-1})<+\infty$ (see Proposition 1), letting $A\to+\infty$ yields

$$R_n^{\rho}\xrightarrow{n\to+\infty}0 \qquad a.s.\quad \forall\rho>0.$$

□

Lemma 3. *Let $a\in(0,1]$, $r\ge0$ and $p>2$ such that $(\mathbf{H_p})$, $(\mathbf{R_a})$, $(\mathbf{S_{a,r}})$ hold and $p/2+a-1>2r$. Assume that $\mathbb{E}\{U_1^{\otimes3}\}=0$ and that $\mathbb{E}\{|U_1|^{2p}\}<+\infty$. Let $f:\mathbb{R}^d\mapsto\mathbb{R}$ satisfying $(\mathbf{C_f^p})$. Then,*
(a) If $\Gamma_n^{(2)}/\sqrt{\Gamma_n}\to0$,

$$\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n\Theta_3(\gamma_k,\bar{X}_{k-1},\bar{X}_{k,2},Z^{(k)})\underset{n\to+\infty}{\xrightarrow{\mathbb{P}}}0.$$

(b) If $\Gamma_n^{(2)}/\sqrt{\Gamma_n}\to\hat{\gamma}\in(0,+\infty]$,

$$\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^n\Theta_3(\gamma_k,\bar{X}_{k-1},\bar{X}_{k,2},Z^{(k)})\underset{n\to+\infty}{\xrightarrow{\mathbb{P}}}\int\phi_3(x)\nu(dx)$$

where ϕ_3 is defined like in Theorem 1.

The proof of this lemma is realized in subsection 4.3.

Remark 9. If Z is a compensated compound Poisson process, computing the rate of convergence of Θ_3 consists in evaluating what happens after its first jump. Naturally, this argument has no sense when the Lévy measure is not finite but the proof and the formulation of ϕ_3 show that it keeps some sense in average.

Lemma 4. *Let $a \in (0,1]$, $r \geq 0$ and $p > 2$ such that $(\mathbf{H_p})$, $(\mathbf{R_a})$, $(\mathbf{S_{a,r}})$ hold and $p/2 + a - 1 > 2r$. Assume that $\mathbb{E}\{U_1^{\otimes 3}\} = 0$ and that $\mathbb{E}\{|U_1|^{2p}\} < +\infty$. Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ satisfying $(\mathbf{C_f^p})$. Then,*
(a) If $\Gamma_n^{(2)}/\sqrt{\Gamma_n} \to 0$,

$$\frac{1}{\sqrt{\Gamma_n}} \sum_{k=1}^{n} \Theta_1(\gamma_k, \bar{X}_{k-1}) + \Theta_2(\gamma_k, \bar{X}_{k-1}, U_k) \xrightarrow[n\to+\infty]{\mathbb{P}} 0,$$

$$\frac{1}{\sqrt{\Gamma_n}} \sum_{k=1}^{n} R_1(\gamma_k, \bar{X}_{k-1}, U_k) + R_2(\gamma_k, \bar{X}_{k-1}, U_k) + R_3(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)}) \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

(b) If $\Gamma_n^{(2)}/\sqrt{\Gamma_n} \xrightarrow{n\to+\infty} \hat{\gamma} \in (0,+\infty]$, we have

$$\frac{1}{\Gamma_n^{(2)}} \sum_{k=1}^{n} \Theta_1(\gamma_k, \bar{X}_{k-1}) \xrightarrow[n\to+\infty]{\mathbb{P}} m_1 \quad \text{and} \quad \frac{1}{\Gamma_n^{(2)}} \sum_{k=1}^{n} \Theta_2(\gamma_k, \bar{X}_{k-1}, U_k) \xrightarrow[n\to+\infty]{\mathbb{P}} m_2,$$

$$\text{with} \quad m_1 = \frac{1}{2}\int D^2 f(x) b(x)^{\otimes 2} \nu(dx)$$

$$\text{and,} \quad m_2 = \int\int \frac{1}{6} D^3 f(x); b(x); (\sigma(x)u)^{\otimes 2} + \frac{1}{24} D^4 f(x)(\sigma(x)u)^{\otimes 4} \mathbb{P}_{U_1}(du)\nu(dx).$$

At last,

$$\frac{1}{\Gamma_n^{(2)}} \sum_{k=1}^{n} R_1(\gamma_k, \bar{X}_{k-1}, U_k) + R_2(\gamma_k, \bar{X}_{k-1}, U_k) + R_3(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)}) \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

Proof. The arguments of this proof are quite similar to those of the previous lemma. Then, we leave it to the reader. □

4.2 Synthesis and proof of Theorem 1

• Proof of Theorem 1 when $\Gamma_n^{(2)}/\sqrt{\Gamma_n} \to 0$: Looking into the decomposition of $\bar{\nu}_n(Af)$ introduced in lemma 1, we deduce from lemmas 2, 3(a) and 4(a) that

$$\sqrt{\Gamma_n}\bar{\nu}_n(Af) - \Big(\frac{f(\bar{X}_n) - f(\bar{X}_0)}{\sqrt{\Gamma_n}}\Big) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}\Big(0, \hat{\sigma}_f^2\Big). \tag{15}$$

Now, $f \leq CV$. Then, by Proposition 1$(a).iii$ and Jensen's inequality, $\mathbb{E}\{f(\bar{X}_n)\} \leq \mathbb{E}\{V^p(\bar{X}_n)\} \leq C\Gamma_n^{\frac{1}{p}}$. It implies that

$$\frac{f(\bar{X}_n) - f(\bar{X}_0)}{\sqrt{\Gamma_n}} \xrightarrow[n\to+\infty]{L^1} 0$$

and Theorem 1 is obvious.
• Proof of Theorem 1 when $\Gamma_n^{(2)}/\sqrt{\Gamma_n} \to \hat{\gamma} \in (0,+\infty]$: in this case, $\sqrt{\Gamma_n} \leq C\Gamma_n^{(2)}$. It implies that

$$\frac{f(\bar{X}_n) - f(\bar{X}_0)}{\Gamma_n^{(2)}} \xrightarrow[n\to+\infty]{L^1} 0.$$

According to Lemmas 2, 3(b) and 4(b), we have

$$\frac{\Gamma_n}{\Gamma_n^{(2)}}\bar{\nu}_n(Af) - \Big(\frac{f(\bar{X}_n) - f(\bar{X}_0)}{\Gamma_n^{(2)}}\Big) \begin{cases} \xrightarrow[n\to+\infty]{\mathbb{P}} m & \text{if } \hat{\gamma} = +\infty \\ \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}\big(\hat{\gamma}m, \hat{\sigma}_f^2\big) & \text{if } \hat{\gamma} < +\infty \end{cases}$$

and the result follows.

4.3 Proof of Lemma 3

In the proof of Lemma 3, we usually need to show that some sequences tend to 0 in probability. The arguments used for this are collected in the following lemma (these arguments also work for the proof of Lemma 4).

Lemma 5. *Let $a \in (0,1]$, $r \geq 0$ and $p > 2$. Assume $(\mathbf{H_p})$, $(\mathbf{R_a})$ and $(\mathbf{S_{a,r}})$. Suppose that $\mathbb{E}\{|U_1|^{2p}\} < +\infty$ and let (F_k) be a sequence of random variables such that F_k is $\mathcal{F}_k$-measurable.*
(a) Assume that $\Gamma_n^{(2)}/\sqrt{\Gamma_n} \to 0$.
i. If $|F_k| \leq C\gamma_k^2 V^{\frac{p}{2}+a-1}(\bar{X}_{k-1})$, then, $1/\sqrt{\Gamma_n}\sum_{k=1}^n F_{k-1} \xrightarrow[n\to+\infty]{\mathbb{P}} 0$.
ii. If $\mathbb{E}\{F_k/\mathcal{F}_{k-1}\} = 0$ and $\mathbb{E}\{|F_k|^2/\mathcal{F}_{k-1}\} \leq C\big(\gamma_k^3 V^p(\bar{X}_{k-1}) + \gamma_k^2 V^{\frac{\epsilon p}{2}}(\bar{X}_{k-1})\big)$ with $\epsilon \in [0,1)$ then, $1/\sqrt{\Gamma_n}\sum_{k=1}^n F_k \xrightarrow[n\to+\infty]{\mathbb{P}} 0$.
(b) Assume that $\Gamma_n^{(2)}/\sqrt{\Gamma_n} \xrightarrow{n\to+\infty} \hat{\gamma} \in (0,+\infty]$. Then,
i. If $|F_k| \leq C\gamma_k^{2+\delta} V^{\frac{p}{2}+a-1}(\bar{X}_{k-1})$, $1/\Gamma_n^{(2)}\sum_{k=1}^n F_{k-1} \xrightarrow[n\to+\infty]{\mathbb{P}} 0$.
ii. If $\mathbb{E}\{F_k/\mathcal{F}_{k-1}\} = 0$ and $\mathbb{E}\{|F_k|^2/\mathcal{F}_{k-1}\} \leq C\big(\gamma_k^3 V^{\epsilon p}(\bar{X}_{k-1}) + \gamma_k^2 V^{\frac{\epsilon p}{2}}(\bar{X}_{k-1})\big)$ with $\epsilon \in [0,1)$ then, $1/\Gamma_n^{(2)}\sum_{k=1}^n F_k \xrightarrow[n\to+\infty]{\mathbb{P}} 0$.

Proof. *(a)* i. By Proposition $1(a).iii$, $\mathbb{E}\{V^p(\bar{X}_n)\} \leq C\Gamma_n$. We then derive from Jensen's inequality that

$$\frac{1}{\sqrt{\Gamma_n}}\mathbb{E}\{\sum_{k=1}^n |F_{k-1}|\} \leq \frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n \gamma_k^2 \Gamma_k^{\frac{\bar{p}}{p}}.$$

where $\bar{p} = p/2 + a - 1$. Hence, the first assertion is obvious if

$$\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n \gamma_k^2 \Gamma_k^{\frac{\bar{p}}{p}} \xrightarrow{n\to+\infty} 0. \tag{16}$$

If (16) is not fulfilled, then we have $\liminf \frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n \gamma_k^2\sqrt{\Gamma_k} > 0$ because $\bar{p}/p \leq 1/2$. It follows from the Kronecker Lemma that we have necessary $\sum_{k\geq 1}\gamma_k^2 = +\infty$. By setting $\eta_k = \gamma_k^2$, we can apply Proposition 1 and deduce that

$$\sup_{n\geq 1}\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^n \gamma_k^2 V^{\frac{p}{2}+a-1}(\bar{X}_{k-1}) < +\infty \quad a.s. \tag{17}$$

Since $\Gamma_n^{(2)}/\sqrt{\Gamma_n} \xrightarrow{n\to+\infty} 0$, the first assertion follows when (16) is not fulfilled.
ii. Since $\mathbb{E}\{V^p(\bar{X}_n)\} \leq \Gamma_n$, we derive from Jensen's inequality that $\mathbb{E}\{|F_k|^2\} \leq C(\gamma_k^3\Gamma_k + \gamma_k^2\Gamma_k^{\frac{\epsilon}{2}})$, with $\epsilon \in [0,1)$. On the one hand, one checks that $\sum_{k=1}^n \gamma_k^3\Gamma_k \leq (\Gamma_n^{(2)})^2$. Hence, since $\Gamma_n^{(2)}/\sqrt{\Gamma_n} \xrightarrow{n\to+\infty} 0$, we have

$$\frac{1}{\Gamma_n}\sum_{k=1}^n \gamma_k^3\Gamma_k \leq C\frac{(\Gamma_n^{(2)})^2}{(\sqrt{\Gamma_n})^2} \xrightarrow{n\to+\infty} 0. \tag{18}$$

On the other hand, one observes that $\sum_{k=1}^n \frac{\gamma_k^2\Gamma_k^{\frac{\epsilon}{2}}}{\Gamma_k} \leq \sum_{k=1}^n \frac{\gamma_k^2}{(\Gamma_k^{(2)})^{2-\epsilon}} < +\infty$. Hence, the Kronecker Lemma implies that

$$\frac{1}{\Gamma_n}\sum_{k=1}^n \gamma_k^2\Gamma_k^{\frac{\epsilon}{2}} \xrightarrow{n\to+\infty} 0. \tag{19}$$

It follows that $\frac{1}{\Gamma_n}\sum_{k=1}^n \mathbb{E}\{|F_k|^2\} \xrightarrow{n\to+\infty} 0$. This yields the second assertion of (a).
(b) i. We derive from the assumptions that

$$\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^n |F_k| \le \frac{C}{\Gamma_n^{(2)}}\sum_{k=1}^n \gamma_k^{2+\delta} V^{\frac{p}{2}+a-1}(\bar{X}_{k-1}). \tag{20}$$

Then, $(b).i.$ follows from (17) which is still valid because $\sum_{k\ge1}\gamma_k^2 = +\infty$.
ii. It suffices to check that

$$\frac{1}{(\Gamma_n^{(2)})^2}\sum_{k=1}^n \mathbb{E}\{|F_k|^2\} \xrightarrow{n\to+\infty} +\infty. \tag{21}$$

With the same arguments as in $(a).ii$, one checks that

$$\frac{1}{(\Gamma_n^{(2)})^2}\sum_{k=1}^n \mathbb{E}\{|F_k|^2\} \le \frac{C}{(\Gamma_n^{(2)})^2}\sum_{k=1}^n \gamma_k^2(\Gamma_k^{(2)})^\epsilon + \frac{C}{(\Gamma_n^{(2)})^2}\sum_{k=1}^n \gamma_k^3\Gamma_k^\epsilon \quad \text{with } \epsilon\in[0,1). \tag{22}$$

On the one hand, we deduce from the Kronecker Lemma that

$$\frac{1}{(\Gamma_n^{(2)})^2}\sum_{k=1}^n \gamma_k^2(\Gamma_k^{(2)})^\epsilon \xrightarrow{n\to+\infty} 0.$$

On the other hand, for every $\epsilon\in[0,1)$

$$\frac{1}{(\Gamma_n^{(2)})^2}\sum_{k=1}^n \gamma_k^3\Gamma_k^\epsilon \overset{n\to+\infty}{=} \frac{1}{(\Gamma_n^{(2)})^2}\sum_{k=1}^n \gamma_k^3\Gamma_k^\epsilon < +\infty$$

because $\sum_{k=1}^n \gamma_k^3\Gamma_k \le (\Gamma_n^{(2)})^2$. Hence, we derive from (22) that

$$\frac{1}{(\Gamma_n^{(2)})^2}\sum_{k=1}^n \mathbb{E}\{|F_k|^2\} \xrightarrow{n\to+\infty} 0$$

which completes the proof. □

Proof of Lemma 3. (a) In order to alleviate the notations, we prove the lemma in the one-dimensional case. We set

$$\bar{\Theta}_3(\gamma,x,z,Z) = \int_0^\gamma ds\int \pi(dy_1)\Big(\tilde{H}^f(z+\kappa(x)Z_s,x,y_1) - \tilde{H}^f(x,x,y_1)\Big).$$

$\bar{\Theta}_3(\gamma,x,z,Z)$ is the compensator of $\Theta_3(\gamma,x,z,Z)$. Then, since $\Theta_3(\gamma,x,z,Z)$ is a purely discontinuous process, we have

$$\mathbb{E}\{|(\Theta_3-\bar{\Theta}_3)(\gamma,x,z,Z)|^2\} = \mathbb{E}\{\int_0^\gamma ds\int \pi(dy_1)\big|\tilde{H}^f(z+\kappa(x)Z_s,x,y_1) - \tilde{H}^f(x,x,y_1)\big|^2\}.$$

By Taylor's formula,

$$\tilde{H}^f(z,x,y_1) = \int_0^1 \big(f'(z+\theta\kappa(x)y_1) - f'(x)\big)\,\kappa(x)y_1 d\theta.$$

It follows that,

$$
\begin{aligned}
|\tilde{H}^f(z+\kappa(x)Z_s,x,y_1)-&\tilde{H}^f(x,x,y_1)| \\
&\leq \sup_{\theta\in[0,1]} |f'(z+\kappa(x)(Z_s+\theta y_1))-f'(x+\theta\kappa(x)y_1)|.|\kappa(x)y_1| \\
&+|f'(z+\kappa(x)Z_s)-f'(x)|.|\kappa(x)y_1|.
\end{aligned}
$$

f' is a Lipschitz continuous function. Then, by setting $z=x+\gamma b(x)+\sqrt{\gamma}\sigma(x)u$, we deduce from the assumptions on the coefficients and from the fact that $\mathbb{E}\{|Z_s|^2\}=O(s)$ that

$$
\begin{aligned}
\mathbb{E}\{|(\Theta_3-&\bar{\Theta}_3)(\gamma,x,z,Z)|^2\} \\
&\leq C\mathbb{E}\{\int_0^\gamma ds\int\pi(dy_1)\Big(\gamma^2|b|^2(x)+\gamma|\sigma(x)|^2u^2+|\kappa(x)|^2|Z_s|^2\Big)|\kappa(x)|^2y_1^2\} \\
&\leq C\big(\gamma^3V^{a+r}(x)+\gamma^2(1+|u|^2)V^{2r}(x)\big). \qquad (23)
\end{aligned}
$$

Set $F_k=(\Theta_3-\bar{\Theta}_3)(\gamma_k,\bar{X}_{k-1},\bar{X}_{k,2},Z^{(k)})$. Since $\mathbb{E}\{F_k/\mathcal{F}_{k-1}\}=0$, $a+r<p$ and $2r<p/2+a-1$, it follows from Lemma 5$(a).ii$ and from the preceding inequality that

$$
\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n(\Theta_3-\bar{\Theta}_3)(\gamma_k,\bar{X}_{k-1},\bar{X}_{k,2},Z^{(k)})\xrightarrow{\mathbb{P}}0 \quad \text{when } n\to+\infty. \qquad (24)
$$

Now, since $f^{(2)}$ is bounded, we deduce from Taylor's formula that

$$
|\tilde{H}^f(z+\kappa(x)Z_s,x,y_1)-\tilde{H}^f(x,x,y_1)|\leq C\|\kappa(x)\|^2.|y_1|^2.
$$

Then,

$$
\mathbb{E}\{|\bar{\Theta}_3(\gamma,x,z,Z)|^2\}\leq C\gamma^2\|\kappa(x)\|^4\int|y_1|^2\pi(dy_1)^2\leq C\gamma^2V^{2r}(x).
$$

By Lemma 5$(a).ii$, it follows that,

$$
\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n\Big(\bar{\Theta}_3(\gamma_k,\bar{X}_{k-1},\bar{X}_{k,2},Z^{(k)})-\mathbb{E}\{\bar{\Theta}_3(\gamma_k,\bar{X}_{k-1},\bar{X}_{k,2},Z^{(k)})/\mathcal{F}_{k-1}\}\Big)\xrightarrow[\mathbb{P}]{n\to+\infty}0. \qquad (25)
$$

Then, by (24) and (25), (a) is obvious if we prove that

$$
\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n\mathbb{E}\{\bar{\Theta}_{3,1}(\gamma_k,\bar{X}_{k-1},\bar{X}_{k,2},Z^{(k)})/\mathcal{F}_{k-1}\}\xrightarrow[n\to+\infty]{\mathbb{P}}0 \qquad \text{and,} \qquad (26)
$$

$$
\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n\mathbb{E}\{\bar{\Theta}_{3,2}(\gamma_k,\bar{X}_{k-1},\bar{X}_{k,2},Z^{(k)})/\mathcal{F}_{k-1}\}\xrightarrow[n\to+\infty]{\mathbb{P}}0 \qquad \text{with} \qquad (27)
$$

$$
\begin{aligned}
&\bar{\Theta}_{3,1}(\gamma,x,z,Z) \\
&=\gamma\bar{\pi}\int_0^1ds\int\tilde{\pi}(dy_1)\int_0^1d\theta\,\Big(f^{(2)}(z+\kappa(x)(Z_{s\gamma}+\theta y_1))-f^{(2)}(z+\theta\kappa(x)y_1)\Big)(1-\theta)\kappa^2(x), \\
&\bar{\Theta}_{3,2}(\gamma,x,z)=\gamma\int\tilde{\pi}(dy_1)\int_0^1d\theta\,\Big(f^{(2)}(z+\theta\kappa(x)y_1)-f^{(2)}(x+\theta\kappa(x)y_1)\Big)(1-\theta)\kappa^2(x)
\end{aligned}
$$

where $\bar{\pi}=\int y_1^2\pi(dy)(<+\infty)$ and $\tilde{\pi}$ is a probability measure defined by $\tilde{\pi}(dy_1)=y_1^2\pi(dy_1)/\bar{\pi}$. Let us prove (26). By Itô's formula, we have

$$
\begin{aligned}
f^{(2)}(z+\kappa(x)(Z_{s\gamma}+\theta y_1))-f^{(2)}(z+\theta\kappa(x)y_1)=&\int_0^{s\gamma}f^{(3)}\big(z+\kappa(x)(Z_{v^-}+\theta y_1)\big)\kappa(x)dZ_v \\
&+\sum_{0<v<s\gamma}\tilde{H}^{f^{(2)}}\big(z+\kappa(x)(Z_{v^-}+\theta y_1),x,\Delta Z_v\big).
\end{aligned}
$$

Since $f^{(3)}$ is bounded, the first term of the right-hand side is a martingale. Therefore, we obtain by the compensation formula that

$$\begin{aligned}
&\mathbb{E}\{\bar{\Theta}_{3,1}(\gamma, x, z, Z)\} \\
&= \gamma\bar{\pi}\int_0^1 ds \int \tilde{\pi}(dy_1)\int_0^1 d\theta \int_0^{s\gamma} dv \int \pi(dy_2)\mathbb{E}\big\{\tilde{H}^{f^{(2)}}(z+\kappa(x)(Z_v+\theta y_1), x, y_2)\big\}(1-\theta)\kappa^2(x).
\end{aligned}$$

Finally, since

$$|\tilde{H}^{f^{(2)}}\big(z+\kappa(x)(Z_v+\theta y_1), x, y_2\big)| \le C\|f^{(4)}\|_\infty \kappa^2(x)y_2^2,$$

we have

$$\mathbb{E}\{|\bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)})|/\mathcal{F}_{k-1}\} \le C\gamma_k^2\kappa^4(\bar{X}_{k-1}) \le C\gamma_k^2 V^{2r}(\bar{X}_{k-1}).$$

Since $2r \le p/2 + a - 1$, (26) follows from Lemma 5$(a).i$.
Now, let us prove (27). Set $z = x + \gamma b(x) + \sqrt{\gamma}\sigma(x)U_1$. By Taylor's formula, there exist

$$\xi_1 \in [x+\sqrt{\gamma}\sigma(x)U_1+\theta\kappa(x)y_1, z+\theta\kappa(x)y_1] \quad \text{and} \quad \xi_2 \in [x+\theta\kappa(x)y_1, x+\sqrt{\gamma}\sigma(x)U_1+\theta\kappa(x)y_1]$$

such that

$$\begin{aligned}
&f^{(2)}(z+\theta\kappa(x)y_1) - f^{(2)}(x+\kappa(x)\theta\kappa(x)y_1) \\
&\qquad = \gamma f^{(3)}(\xi_2)b(x) + \sqrt{\gamma}f^{(3)}(x+\theta\kappa(x)y_1)\sigma(x)U_1 + \gamma f^{(4)}(\xi_1)\sigma^2(x)U_1^2.
\end{aligned}$$

Since $f^{(3)}$ and $f^{(4)}$ are bounded and U_1 is centered,

$$|\mathbb{E}\{\bar{\Theta}_{3,2}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2})/\mathcal{F}_{k-1}\}| \le C\gamma_k^2\big(|b|\kappa^2(\bar{X}_{k-1}) + \sigma^2\kappa^2(\bar{X}_{k-1})\big) \le C\gamma_k^2 V^{a\vee 2r}(\bar{X}_{k-1})$$

where we have used that $a/2 + r \le a \vee 2r$. Since $a \vee 2r \le p/2 + a - 1$, we deduce (27) from Lemma 5(a).i.

$b)$ We keep the notations of (a). On the one hand, by (23) and Lemma 5$(b).ii$, we have

$$\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^n (\Theta_3 - \bar{\Theta}_3)(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)}) \xrightarrow[\mathbb{P}]{n\to+\infty} 0.$$

On the other hand, we have

$$\mathbb{E}\{|\bar{\Theta}_3(\gamma, x, z, Z)|^2\} \le C\gamma^2 V^{\frac{\epsilon p}{2}}(x)$$

with $\epsilon \in [0,1)$. Hence, by Lemma 5$(b).ii$, it follows that

$$\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^n \Big(\bar{\Theta}_3(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)}) - \mathbb{E}\{\bar{\Theta}_3(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)})/\mathcal{F}_{k-1}\}\Big) \xrightarrow[\mathbb{P}]{n\to+\infty} 0.$$

Finally, it suffices to prove that

$$\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^n \mathbb{E}\{\bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)})/\mathcal{F}_{k-1}\} \xrightarrow[n\to+\infty]{\mathbb{P}} m_{3,1} \tag{28}$$

$$\text{and,} \quad \frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^n \mathbb{E}\{\bar{\Theta}_{3,2}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)})/\mathcal{F}_{k-1}\} \xrightarrow[n\to+\infty]{\mathbb{P}} m_{3,2}. \tag{29}$$

with $m_{3,1} + m_{3,2} = \int \phi_3(x)\nu(dx)$. In order to prove (28), we first show that

$$\frac{1}{\Gamma_n^{(2)}} \sum_{k=1}^{n} \Big(\mathbb{E}\{\bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)})/\mathcal{F}_{k-1}\} - \bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k-1}, 0) \Big) \xrightarrow[n\to+\infty]{\mathbb{P}} 0. \quad (30)$$

Since $f^{(4)}$ is a bounded and Lipschitz continuous function, $f^{(4)}$ is also 2δ-Holder for every $\delta \in (0, 1/2]$, *i.e.*

$$[f^{(4)}]_{2\delta} = \sup_{x,y\in\mathbb{R}^d} \frac{|f^{(4)}(y) - f^{(4)}(x)|}{|y-x|^{2\delta}} < +\infty.$$

It follows from the Taylor formula that

$$\Big|\tilde{H}^{f^{(2)}}(z + \kappa(x)(Z_v + \theta y_1), x, y_2) - \tilde{H}^{f^{(2)}}(z + \kappa(x)(Z_v + \theta y_1), x, y_2)\Big| \\ \leq C[f^{(4)}]_{2\delta}\Big(|z-x|^{2\delta} + \kappa(x)^{2\delta}|Z_v|^{2\delta}\Big)\kappa(x)^2 y_2^2.$$

By setting $z = x + \gamma b(x) + \sqrt{\gamma}\sigma(x)u$ and taking δ sufficiently small, we have

$$\mathbb{E}\{|\tilde{H}^{f^{(2)}}(z+\kappa(x)(Z_v+\theta y_1), x, y_2) - \tilde{H}^{f^{(2)}}(x+\kappa(x)\theta y_1, x, y_2)|\} \leq C\gamma^{\delta}(1+|u|^{2\delta})V^{\frac{p}{2}+a-1}(x).$$

This implies that

$$\Big|\mathbb{E}\Big\{\bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)}) - \bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k-1}, 0)/\mathcal{F}_{k-1}\Big\}\Big| \leq C\gamma_k^{2+\delta}V^{\frac{p}{2}+a-1}(\bar{X}_{k-1}).$$

Then, (30) follows from Lemma 5$(b).i$.
Now, $\bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k-1}, 0)$ is $\mathcal{F}_{k-1}$-measurable and

$$|\bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k-1}, 0)| \leq C\gamma_k^2 V^{2r}(\bar{X}_{k-1}).$$

Since $2r < p/2 + a - 1$, we can apply Proposition 1 with $\eta_k = \gamma_k^2$. We obtain

$$\frac{1}{\Gamma_n^{(2)}} \sum_{k=1}^{n} \bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k-1}, 0) = \frac{1}{\Gamma_n^{(2)}} \sum_{k=1}^{n} \frac{\gamma_k^2}{2}\phi_{3,1}(\bar{X}_{k-1}) \xrightarrow{n\to+\infty} \int \phi_{3,1}(x)\nu(dx) \quad a.s.$$

$$\text{with} \quad \phi_{3,1}(x) = \frac{1}{2}\int \pi(dy_1)\int_0^1 d\theta \int \pi(dy_2)\tilde{H}^{f^{(2)}}\big(x + \kappa(x)\theta y_1, x, y_2\big)(1-\theta)\kappa^2(x)y_1^2 \\ = \frac{1}{2}\int \pi(dy_1)\int \pi(dy_2)\tilde{H}^{\tilde{H}^f_{\cdot,x,y_1}}(x, x, y_2).$$

It follows from (30) that

$$\frac{1}{\Gamma_n^{(2)}} \sum_{k=1}^{n} \bar{\Theta}_{3,1}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2}, Z^{(k)})/\mathcal{F}_{k-1}\} \xrightarrow{n\to+\infty} m_{3,1} = \int \phi_{3,1}(x)\nu(dx) \quad a.s.$$

Finally, we prove (29). Since $f^{(3)}$ and $f^{(4)}$ are bounded and Lipschitz continuous, we deduce from Taylor's formula that for every $\delta \in [0, 1/2]$,

$$\mathbb{E}\{\bar{\Theta}_{3,2}(\gamma_k, \bar{X}_{k-1}, \bar{X}_{k,2})/\mathcal{F}_{k-1}\} = \gamma_k^2(\phi_{3,2}(\bar{X}_{k-1}) + \phi_{3,3}(\bar{X}_{k-1})) + \bar{\rho}_1(\bar{X}_{k-1}, \gamma_k) + \bar{\rho}_2(\bar{X}_{k-1}, \gamma_k)$$

$$\text{with} \quad \phi_{3,2}(x) = \int \pi(dy_1)\int_0^1 d\theta f^{(3)}(x + \theta\kappa(x)y_1)b(x)(1-\theta)(\kappa(x)y_1)^2$$

$$\phi_{3,3}(x) = \int \pi(dy_1)\int_0^1 d\theta \int \mathbb{P}_{U_1}(du)f^{(4)}(x + \theta\kappa(x)y_1)\sigma^2(x)u^2(1-\theta)(\kappa(x)y_1)^2$$

$$|\bar{\rho}_1(x,\gamma)| \leq [f^{(3)}]_{\delta}\gamma^{2+\delta}|b(x)|^{1+\delta}|\kappa(x)|^2 \quad \text{and} \quad |\bar{\rho}_2(x,\gamma)| \leq [f^{(4)}]_{2\delta}\gamma^{2+\delta}|\sigma(x)|^{2(1+\delta)}|\kappa(x)|^2.$$

Since $(a/2+r) \vee (2r) < p/2+a-1$, one can find $\delta > 0$ such that

$$|b(x)|^{1+\delta}|\kappa(x)|^2 + |\sigma(x)|^{2(1+\delta)}|\kappa(x)|^2 \leq V^{\frac{p}{2}+a-1}(x).$$

On the one hand, Lemma $5(b).i$ and the assumptions on the coefficients allow us to conclude that

$$\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^{n}\Big(\bar{\rho}_1(\bar{X}_{k-1},\gamma_k)+\bar{\rho}_2(\bar{X}_{k-1},\gamma_k)\Big) \xrightarrow{n\to+\infty} 0.$$

On the other hand, as $|b|(x)|\kappa|^2(x) + |\sigma|^2(x)|\kappa|^2(x) = o(V^{\frac{p}{2}+a-1}(x))$ when $|x| \to +\infty$, we derive from Proposition 1 applied with $\eta_k = \gamma_k^2$ that

$$\frac{1}{\Gamma_n^{(2)}}\sum_{k=1}^{n}\gamma_k^2(\phi_{3,2}(\bar{X}_{k-1})+\phi_{3,3}(\bar{X}_{k-1})) \xrightarrow{n\to+\infty} m_{3,2}$$

with $m_{3,2} = \int(\phi_{3,2}(x)+\phi_{3,3}(x))\nu(dx)$. Checking that

$$\phi_{3,2}(x)+\phi_{3,3}(x) = \int \pi(dy_1)\Big((H^f_{.,x,y_1})'(x)b(x) + \int \mathbb{P}_{U_1}(du)(H^f_{.,x,y_1})^{(2)}(x)(\sigma(x)u)^2\Big).$$

completes the proof.

5 Proof of Theorem 2

The proof is built as follows. Like in the proof of Theorem 1, we firstly decompose $\bar{\nu}_n^{P}(Af)$ and $\bar{\nu}_n^{W}(Af)$ (see Lemma 6). Some new terms appear due to the approximation of the jump component. That is why in the sequel, we focus on these parts of the decomposition (see Lemmas 7 and 8). The other terms can be studied by the same process as their corresponding terms in the decomposition of $\bar{\nu}_n(Af)$ and then, are left to the reader. We denote by $(Z^{(k)_P})_{k\geq 1}$, a sequence of independent and càdlàg processes such that

$$(Z_t^{(k)_P})_{t\geq 0} \stackrel{\mathcal{L}}{=} (Z_{t,k})_{t\geq 0} \quad \text{and} \quad Z_{\gamma_k}^{(k)_P} = \bar{Z}_k^{P} \quad \forall k \geq 1$$

For a $\mathcal{C}^2$-function f such that $D^2 f$ is bounded, we define $A^{k,P}$ and $A^{k,W}$ by

$$A^{k,P}f(x) = \langle \nabla f, b\rangle(x) + \frac{1}{2}\mathrm{Tr}(\sigma^* D^2 f\sigma)(x) + \int_{\{|y|>u_k\}} \tilde{H}^f(x,x,y)\pi(dy)$$

$$A^{k,W}f(x) = A^{k,P}f(x) + \frac{1}{2}\int_{\{|y|\leq u_k\}} D^2 f(x)(\kappa(x)y)^{\otimes 2}\pi(dy),$$

These operators correspond respectively to the infinitesimal generators of

$$dX_t = b(X_{t^-})dt + \sigma(X_{t^-})dW_t + \kappa(X_{t^-})dZ_{t,k} \quad \text{and}$$
$$dX_t = b(X_{t^-})dt + \sigma(X_{t^-})dW_t + \kappa(X_{t^-})d(Z_{t,k} + Q_k\tilde{W}_t),$$

where $\tilde{W}$ is a q-dimensional Brownian motion independent of W and $(Z_{t,k})_{t\geq 0}$.

Lemma 6. *For a C^2-function f such that $D^2 f$ is bounded, we have the following decompositions*

$$1) \quad \sum_{k=1}^{n}\gamma_k Af(\bar{X}_{k-1}^{P}) = G_n^{P} + f(\bar{X}_n^{P}) - f(x) - \sum_{k=1}^{n}\Big(\xi_1(\gamma_k,\bar{X}_{k-1}^{P},U_k) + \xi_{2,P}^k(\gamma_k,\bar{X}_{k-1}^{P},Z^{(k)_P})\Big)$$
$$-\sum_{k=1}^{n}\Big(\Theta_1(\gamma_k,\bar{X}_{k-1}^{P}) + \Theta_2(\gamma_k,\bar{X}_{k-1}^{P},U_k) + \Theta_3(\gamma_k,\bar{X}_{k-1}^{P},\bar{X}_{k,2}^{P},Z^{(k)_P})\Big)$$
$$-\sum_{k=1}^{n}\Big((R_1+R_2)(\gamma_k,\bar{X}_{k-1}^{P},U_k) + R_3(\gamma_k,\bar{X}_{k-1}^{P},\bar{X}_{k,2}^{P},Z^{(k)_P})\Big),$$

where $\bar{X}^{P}_{k,2} = \bar{X}^{P}_{k-1} + \gamma_k b(\bar{X}^{P}_{k-1}) + \sqrt{\gamma_k}\sigma(\bar{X}^{P}_{k-1})U_k$, $G^{P}_n = \sum_{k=1}^n \gamma_k (Af - A^{k,P} f)(\bar{X}^{P}_{k-1})$ *and for a càdlàg process* Y,

$$\xi^k_{2,P}(\gamma, x, Y) = \int_0^\gamma \langle \nabla f(x), \kappa(x) dY_s\rangle + \Big(\sum_{0<s\leq\gamma} \tilde{H}^f(x, x, \Delta Y_s)) - \gamma \int_{D_k} \tilde{H}^f(x, x, y)\pi(dy)\Big).$$

$$\begin{aligned}
2) \sum_{k=1}^n \gamma_k Af(\bar{X}^{W}_{k-1}) = G^{W}_n + J^{W}_n + f(\bar{X}^{W}_n) - f(x) - \sum_{k=1}^n \Big(\xi_1(\gamma_k, \bar{X}^{W}_{k-1}, U_k) + \xi^k_{2,B}(\gamma_k, \bar{X}^{W}_{k-1}, Z^{(k)_P})\Big)\\
- \sum_{k=1}^n \Big(\Theta_1(\gamma_k, \bar{X}^{W}_{k-1}) + \Theta_2(\gamma_k, \bar{X}^{W}_{k-1}, U_k) + \Theta_3(\gamma_k, \bar{X}^{W}_{k-1}, \bar{X}^{P}_{k,2}, Z^{(k)_P})\Big)\\
- \sum_{k=1}^n \Big((R_1 + R_2)(\gamma_k, \bar{X}^{W}_{k-1}, U_k) + R_3(\gamma_k, \bar{X}^{W}_{k-1}, \bar{X}^{P}_{k,2}, Z^{(k)_P})\Big),
\end{aligned}$$

where $\bar{X}^{W}_{k,2} = \bar{X}^{W}_{k-1} + \bar{X}^{W}_{k-1} + \gamma_k b(\bar{X}^{W}_{k-1}) + \sqrt{\gamma_k}\sigma(\bar{X}^{W}_{k-1})U_k$,

$$G^{W}_n = \sum_{k=1}^n \gamma_k (Af - A^{k,W} f)(\bar{X}^{W}_{k-1}),\ J^{W}_n = -\sum_{k=1}^n f(\bar{X}^{W}_k) - f(\bar{X}^{W}_{k,3}) + \gamma_k (A^{k,P} f - A^{k,W} f)(\bar{X}^{W}_{k-1}),$$

with $\bar{X}^{W}_{k,3} = \bar{X}^{W}_k - \kappa(\bar{X}^{W}_{k-1})Q_k\Lambda_k$.

The two following lemmas are devoted to the additional terms of the preceding decomposition. In Lemma 7, we compute the rate of G^{P}_n and G^{W}_n and in Lemma 8, we show that J^{W}_n does not have any consequences on the rate of the procedure.

Lemma 7. *Let* $a \in (0, 1]$, $r \geq 0$ *and* $p > 2$ *such that* $(\mathbf{H_p})$, $(\mathbf{R_a})$, $(\mathbf{S_{a,r}})$ *hold and* $2r < p/2 + a - 1$. *Suppose that* $\mathbb{E}\{|U_1|^{2p}\} < +\infty$ *and* $\mathbb{E}\{|\Lambda_1|^{2p}\} < +\infty$. *Let* $f : \mathbb{R}^d \mapsto \mathbb{R}$ *satisfying* $(\mathbf{C^p_f})$. *Then,*

(1) i. If $\lim_{n\to+\infty} \beta^{(2)}_{n,\pi} < +\infty$, $\frac{1}{\sqrt{\Gamma_n}} \sum_{k=1}^n \gamma_k (Af - A^{k,P} f)(\bar{X}^{P}_{k-1}) \xrightarrow[n\to+\infty]{\mathbb{P}} 0$.

ii. If $\lim_{n\to+\infty} \beta^{(2)}_{n,\pi} = +\infty$,

$$\limsup_{n\to+\infty} \frac{1}{\beta^{(2)}_{n,\pi}} \sum_{k=1}^n \gamma_k \big|(Af - A^{k,P} f)(\bar{X}^{P}_{k-1})\big| \leq \bar{m}_2 \quad a.s.$$

where $\bar{m}_2 = \frac{\|D^2 f\|_\infty}{2} \int \|\kappa\|^2(x)\nu(dx)$. *Furthermore, if* $(\mathbf{A^2_2})$ *holds,*

$$\frac{1}{\beta^{(2)}_{n,\pi}} \sum_{k=1}^n \gamma_k (Af - A^{k,P} f)(\bar{X}^{P}_{k-1}) \xrightarrow{n\to+\infty} m_2 \quad a.s. \quad \text{with } |m_2| \leq \bar{m}_2. \tag{31}$$

(2) Assume that $s = 3$ *or that* $s = 4$ *if* π *is quasi-symmetric in the neighborhood of 0.*

i. If $\lim_{n\to+\infty} \beta^{(s)}_{n,\pi} < +\infty$ $\frac{1}{\sqrt{\Gamma_n}} \sum_{k=1}^n \gamma_k (Af - A^{k,W} f)(\bar{X}^{W}_{k-1}) \xrightarrow[n\to+\infty]{\mathbb{P}} 0$.

ii. If $\lim_{n\to+\infty} \beta^{(s)}_{n,\pi} = +\infty$,

$$\limsup_{n\to+\infty} \frac{1}{\beta^{(3)}_{n,\pi}} \sum_{k=1}^n \gamma_k \Big|(Af - A^{k,W} f)(\bar{X}^{W}_{k-1})\Big| \leq \bar{m}_s \quad a.s.$$

where $\bar{m}_s = C_s \|D^s f\|_\infty \int \|\kappa(x)\|^s \nu(dx)$ *with* $C_3 = \frac{d^{\frac{3}{2}}}{6}$ *and* $C_4 = \frac{d^2}{24}$. *Furthermore, if ,* $(\mathbf{A^2_s})$ *holds,*

$$\frac{1}{\beta^{(s)}_{n,\pi}} \sum_{k=1}^n \gamma_k (Af - A^{k,W} f)(\bar{X}^{W}_{k-1}) \xrightarrow{n\to+\infty} m_s \quad a.s. \quad \text{with } |m_s| \leq \bar{m}_s. \tag{32}$$

Proof. (1)i. By Taylor's formula, we have

$$|Af(x)-A^{k,P}f(x)| = \frac{1}{2}\int D^2f(\xi_y)(\kappa(x)y)^{\otimes 2}\pi(dy) \leq \frac{\|D^2f\|_\infty}{2}\int\sum_{i,j}|(\kappa(x)y)_i(\kappa(x)y)_j|\pi(dy).$$

For $z\in\mathbb{R}^d$, $|\sum_{i,j}z_iz_j|\leq|z|_1^2\leq d|z|^2$. It follows that

$$|Af(\bar{X}^P_{k-1}) - A^{k,P}f(\bar{X}^P_{k-1})| \leq \frac{d}{2}\|D^2f\|_\infty\|\kappa(\bar{X}^P_{k-1})\|^2\int_{|y|\leq u_k}|y|^2\pi(dy). \tag{33}$$

Since $r\leq p/2$ and $\mathbb{E}\{V^p(\bar{X}^P_{k-1})\}\leq\Gamma_k$, we deduce that

$$\sum_{k=1}^n\gamma_k\mathbb{E}\{|Af(\bar{X}^P_{k-1}) - A^{k,P}f(\bar{X}^P_{k-1})|\} \leq \sum_{k=1}^n\gamma_k\int_{|y|\leq u_k}|y|^2\pi(dy)\sqrt{\Gamma_k}.$$

Now, as $\lim_{n\to+\infty}\beta^{(2)}_{n,\pi}<+\infty$, the Kronecker Lemma yields

$$\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^n\gamma_k\int_{|y|\leq u_k}|y|^2\pi(dy)\sqrt{\Gamma_k}\xrightarrow{n\to+\infty}0.$$

The first assertion is obvious.
ii. Since $\lim_{n\to+\infty}\beta^{(2)}_{n,\pi}=+\infty$, we deduce from Proposition 1 with $\eta_k=\gamma_k\int_{|y|\leq u_k}|y|^2\pi(dy)$ that

$$\frac{1}{\beta^{(2)}_{n,\pi}}\sum_{k=1}^n\gamma_k\int_{|y|\leq u_k}|y|^2\pi(dy)\|\kappa(\bar{X}^P_{k-1})\|^2\xrightarrow{n\to+\infty}\int\|\kappa(x)\|^2\nu(dx)\quad a.s.$$

because $\|\kappa\|^2=o(V^{\frac{p}{2}+a-1})$. Then, the second assertion follows from (33).
Assume now that $(\mathbf{A_2^2})$ holds. Since D^2f is Lipschitz continuous, we deduce from Taylor's formula that

$$Af(x)-A^{k,P}f(x)=\frac{1}{2}\Big(\sum_{i,j}\rho_k(i,j)\psi_{i,j}(x)+R^k_{i,j}(x)\Big)$$

$$\text{with}\quad\rho_k(i,j)=\int_{\{|y|\leq u_k\}}y_iy_j\pi(dy),\quad\psi_{i,j}(x)=\sum_{l,m}\kappa_{i,l}\frac{\partial^2f}{\partial_l\partial_m}\kappa_{m,j}(x))$$

and $|R^k_{i,j}(x)|\leq\int_{\{|y|\leq u_k\}}|y|^3\pi(dy)\|\kappa(x)\|^3$.
According to $(\mathbf{A_2^2})$, for every i,j, $\lim\rho_k(i,j)/\int_{\{|y|\leq u_k\}}|y|^2\pi(dy)=\alpha_{i,j}\in\mathbb{R}$. Set $\eta_k=\gamma_k\int_{\{|y|\leq u_k\}}|y|^2\pi(dy)$ and $H_n=\sum_{k=1}^n\eta_k$. Then,

$$\frac{1}{\beta^{(2)}_{n,\pi}}\sum_{k=1}^n\gamma_k\rho_k(i,j)\psi_{i,j}(\bar{X}^P_{k-1})=\frac{\alpha_{i,j}}{H_n}\sum_{k=1}^n\eta_k\psi_{i,j}(\bar{X}^P_{k-1})+\frac{1}{H_n}\sum_{k=1}^n\varepsilon^1_k\eta_k\psi_{i,j}(\bar{X}^P_{k-1})$$

with $\varepsilon^1_k=(\rho_k(i,j)-\alpha_{i,j}\eta_k)/\eta_k$. Firstly, since $\psi_{i,j}\leq CV^r$ and $r<p/2+a-1$, Proposition 1 applied with $\eta_k=\gamma_k\int_{\{|y|\leq u_k\}}|y|^2\pi(dy)$ yields

$$\frac{1}{H_n}\sum_{k=1}^n\eta_k\psi_{i,j}(\bar{X}^P_{k-1})\xrightarrow{n\to+\infty}\int\psi_{i,j}(x)\nu(dx)\quad a.s.$$

Secondly, $\varepsilon^1_k=o(1)$. Since $\sup_{n\geq1}1/H_n\sum_{k=1}^n\eta_kV^{\frac{p}{2}+a-1}(\bar{X}^P_{k-1})<+\infty$ $a.s.$, it is then easy to check that

$$\frac{1}{H_n}\sum_{k=1}^n\varepsilon^1_k\eta_k\psi_{i,j}(\bar{X}^P_{k-1})\xrightarrow{n\to+\infty}0\quad a.s.$$

The same argument is appropriate for $R^k_{i,j}$ with $\varepsilon_k^2 = (\int_{\{|y|\le u_k\}} |y|^3\pi(dy))/(\int_{\{|y|\le u_k\}} |y|^2\pi(dy))$. Finally, we obtain

$$\frac{1}{\beta^{(2)}_{n,\pi}}\sum_{k=1}^{n}\gamma_k(Af - A^{k,P}f)(\bar{X}^P_{k-1}) \xrightarrow{n\to+\infty} m_2 = \sum_{i,j}\frac{\alpha_{i,j}}{2}\int \psi_{i,j}(x)\nu(dx).$$

2) We derive from Taylor's formula

$$|Af(x) - A^{k,W}f(x)| \le C_s\|D^s f\|_\infty \|\kappa(x)\|^s \int_{|y|\le u_k} |y|^s\pi(dy). \tag{34}$$

with $s = 3$ and $C_3 = \frac{d^{\frac{3}{2}}}{6}$, or $s = 4$ and $C_4 = \frac{d^2}{24}$ if $\int_{|y|\le u_k} y^{\otimes 3}\pi(dy) = 0$. As $2r \le p/2$ and $\mathbb{E}\{V^p(\bar{X}^W_{k-1})\} \le \Gamma_k$, it implies that

$$\sum_{k=1}^{n}\gamma_k\mathbb{E}\{|Af - A^{k,W}f|(\bar{X}^W_{k-1})\} \le C\sum_{k=1}^{n}\gamma_k\int_{|y|\le u_k}|y|^s\pi(dy)\sqrt{\Gamma_k},$$

with $s = 3$ or $s = 4$ if $\int_{|y|\le u_k} y^{\otimes 3}\pi(dy) = 0$. If $\lim_{n\to+\infty}\beta^{(s)}_{n,\pi} < +\infty$, we derive from the Kronecker Lemma that

$$\frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^{n}\gamma_k\int_{|y|\le u_k}|y|^s\pi(dy)\sqrt{\Gamma_k} \xrightarrow{n\to+\infty} 0.$$

and the first assertion of (2) follows.
Assume now that $\lim_{n\to+\infty}\beta^{(s)}_{n,\pi} = +\infty$. Applying Proposition 1 to $f(x) = \|\kappa(x)\|^s$ with $\eta_k = \gamma_k\int_{|y|\le u_k}|y|^s\pi(dy)$ yields

$$\limsup_{n\to+\infty}\frac{1}{\beta^{(s)}_{n,\pi}}\sum_{k=1}^{n}\gamma_k\int_{|y|\le u_k}|y|^s\pi(dy)\|\kappa(\bar{X}^W_{k-1})\|^s < +\infty \quad a.s.$$

Then, (2).ii follows from (34).
Finally, the proof of (32) is similar to that of (31). □

Lemma 8. *Let $a \in (0,1]$, $r \ge 0$ and $p > 2$ such that $(\mathbf{H_p})$, $(\mathbf{R_a})$, $(\mathbf{S_{a,r}})$ hold and $2r < p/2+a-1$. Suppose that $\mathbb{E}\{|U_1|^{2p}\} < +\infty$ and that $\mathbb{E}\{|\Lambda_1|^{2p}\} < +\infty$. Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ satisfying $(\mathbf{C^p_f})$. Then,*

$$\frac{1}{\sqrt{\Gamma_n \vee \Gamma^{(2)}_n}}\sum_{k=1}^{n}\Big(f(\bar{X}^W_k) - f(\bar{X}^W_{k,3}) + \gamma_k(A^{k,P}f - A^{k,W}f)(\bar{X}^W_{k-1})\Big) \xrightarrow[n\to+\infty]{\mathbb{P}} 0. \tag{35}$$

Proof. Since

$$(A^{k,W}f - A^{k,P}f)(x) = \frac{1}{2}\int_{\{|y|\le u_k\}} D^2f(x)(\kappa(x)y)^{\otimes 2}\pi(dy) = \frac{1}{2}\mathbb{E}\{D^2f(x)(\kappa(x)Q_k\Lambda_k)^{\otimes 2}\},$$

we derive from Taylor's formula that

$$\begin{aligned} f(\bar{X}^W_k) - f(\bar{X}^W_{k,3}) + \gamma_k(A^{k,P}f - A^{k,W})f(\bar{X}^W_{k-1}) &= \xi_1(\gamma_k, \bar{X}^W_{k,3}, \bar{X}^W_{k-1}, Q_k\Lambda_k) \\ &+ \tilde{R}_{k,1}(\gamma_k, \bar{X}^W_{k-1}, Q_k\Lambda_k) + \tilde{R}_{k,2}(\gamma_k, \bar{X}^W_{k,3}, \bar{X}^W_{k-1}, Q_k\Lambda_k) \end{aligned}$$

where, $\xi_1(\gamma,z,x,Q_k\Lambda_k)=\sqrt{\gamma}\langle\nabla f(z),\kappa(x)Q_k\Lambda_k\rangle$,
$\tilde{R}_{k,1}(\gamma,x,Q_k\Lambda_k)=\dfrac{\gamma}{2}\big(D^2f(x)(\kappa(x)Q_k\Lambda_k)^{\otimes 2}-\mathbb{E}\{D^2f(x)(\kappa(x)Q_k\Lambda_k)^{\otimes 2}\}\big)$
$\tilde{R}_{k,2}(\gamma,z,x,Q_k\Lambda_k)=\gamma\displaystyle\int_0^1\big(D^2f(z+\theta\sqrt{\gamma}\kappa(x)Q_k\Lambda_k)-D^2f(x)\big)(1-\theta)(\kappa(x)Q_k\Lambda_k)^{\otimes 2}d\theta.$

Setting $\theta_n=\sqrt{\Gamma_n}\vee\Gamma_n^{(2)}$, it suffices to show the three following steps:

a) $\theta_n^{-1}\sum_{k=1}^n\xi_1(\gamma_k,\bar{X}^W_{k-1},Q_k\Lambda_k)\xrightarrow[n\to+\infty]{\mathbb{P}}0$,
b) $\theta_n^{-1}\sum_{k=1}^n\tilde{R}_{k,1}(\gamma_k,\bar{X}^W_{k-1},Q_k\Lambda_k)\xrightarrow[n\to+\infty]{\mathbb{P}}0$.
c) $\theta_n^{-1}\sum_{k=1}^n\tilde{R}_{k,2}(\gamma_k,\bar{X}^W_{k,3},\bar{X}^W_{k-1},Q_k\Lambda_k)\xrightarrow[n\to+\infty]{\mathbb{P}}0$.

a) We set

$$\xi_1(\gamma,z,x,Q_k\Lambda_k)=\xi_{1,1}(\gamma,x,Q_k\Lambda_k)+\xi_{1,2}(\gamma,z,x,Q_k\Lambda_k)$$

with $\xi_{1,1}(\gamma,x,v)=\sqrt{\gamma}\langle\nabla f(x),\kappa(x)v\rangle$ and $\xi_{1,2}(\gamma,z,x,v)=\sqrt{\gamma}\langle\nabla f(z)-\nabla f(x),\kappa(x)v\rangle$. Let $(M_{n,1})$ and $(M_{n,2})$ be the $(\mathcal{F}_n)$-martingales defined by

$$M_{n,1}=\sum_{k=1}^n\xi_{1,1}(\gamma_k,\bar{X}^W_{k-1},Q_k\Lambda_k)\quad\text{and}\quad M_{n,2}=\sum_{k=1}^n\xi_{1,2}(\gamma_k,\bar{X}^W_{k,3},\bar{X}^W_{k-1},Q_k\Lambda_k).$$

We have to prove that $\theta_n^{-1}M_{n,1}\xrightarrow{n\to+\infty}0$ and $\theta_n^{-1}M_{n,2}\xrightarrow{n\to+\infty}0$.
According to $(\mathbf{C^p_f})$ and the assumptions on κ, we check that

$$\frac{<M>_{n,1}}{\Gamma_n}\leq\frac{C}{\Gamma_n}\sum_{k=1}^n\gamma_k\int_{|y|\leq u_k}|y|^2\pi(dy)V^{\frac{p}{2}+a-1}(\bar{X}^W_{k-1}).$$

Now, by (6), $\sup_{n\geq1}1/\Gamma_n\sum_{k=1}^n\gamma_kV^{\frac{p}{2}+a-1}(\bar{X}^W_{k-1})<+\infty$ *a.s.* Since $\int_{|y|\leq u_k}|y|^2\pi(dy)\to0$, that $<M>_{n,1}/\Gamma_n\xrightarrow{n\to+\infty}0$ *a.s.* Then,

$$\frac{1}{\sqrt{\Gamma_n}\vee\Gamma_n^{(2)}}|M_{n,1}|\leq\frac{1}{\sqrt{\Gamma_n}}|M_{n,1}|\xrightarrow[n\to+\infty]{\mathbb{P}}0.\tag{36}$$

Now, we turn to $(M_{n,2})$. Since ∇f is Lipschitz continuous, it follows from the assumptions on the coefficients that

$$\mathbb{E}\{|\xi_{1,2}(\gamma_k,\bar{X}^W_{k,3},\bar{X}^W_{k-1},Q_k\Lambda_k)|^2/\mathcal{F}_{k-1}\}\leq C\int_{|y|\leq u_k}|y|^2\pi(dy)\Big(\gamma_k^3V^{\epsilon p}(\bar{X}^W_{k-1})+\gamma_k^2V^{\epsilon\frac{p}{2}}(\bar{X}^W_{k-1})\Big).$$

with $\epsilon<1$ and $\bar{p}=p/2+a-1$. Then, by a variant of Lemma $5(a).ii$ and $(b).ii$, one checks that

$$\frac{<M_{n,2}>}{(\sqrt{\Gamma_n}\vee\Gamma_n^{(2)})^2}\xrightarrow{n\to+\infty}0\quad a.s.\Longrightarrow\quad\frac{1}{\sqrt{\Gamma_n}\vee\Gamma_n^{(2)}}M_{n,2}\xrightarrow[L^2]{n\to+\infty}0.$$

b) $\tilde{R}_{k,1}$ is very closed to R_2 introduced in Lemma 1 and the arguments are similar.
c) As D^2f is bounded, one observes that

$$\mathbb{E}\{|\tilde{R}_{k,2}(\gamma_k,\bar{X}^W_{k,2},\bar{X}^W_{k-1},Q_k\Lambda_k)|^2/\mathcal{F}_{k-1}\}\leq C\gamma_k^2V^{2r}(\bar{X}^W_{k-1}).$$

Then, since $2r < p/2$, a variant of Lemma $5(a).ii$ and $(b).ii$ yields

$$\frac{1}{\sqrt{\Gamma_n \vee \Gamma_n^{(2)}}} \sum_{k=1}^{n} \Big(\tilde{R}_{k,2}(\gamma_k, \bar{X}^W_{k,2}, \bar{X}^W_{k-1}, Q_k\Lambda_k) - \mathbb{E}\{\tilde{R}_{k,2}(\gamma_k, \bar{X}^W_{k,2}, \bar{X}^W_{k-1}, Q_k\Lambda_k)/\mathcal{F}_{k-1}\} \Big) \xrightarrow[n\to+\infty]{\mathbb{P}} 0. \tag{37}$$

By setting $\zeta_{k,\theta}(x) = x + \gamma_k b(x) + \sqrt{\gamma_k}\sigma(x)U_k + \kappa(x)(\bar{Z}_k + \theta Q_k \Lambda_k)$, we decompose the integrand of $\tilde{R}_{k,2}$ as follows:

$$D^2 f(\zeta_{k,\theta}(x)) - D^2 f(x) = \big(D^2 f(x+\gamma_k b(x)) - D^2 f(x)\big) + \big(D^2 f(\zeta_{k,\theta}(x)) - D^2 f(x+\gamma_k b(x))\big).$$

On the one hand, set $y_k = \sqrt{\gamma_k} Q_k \Lambda_k$. By Taylor's formula, we have

$$(D^2 f(x+\gamma_k b(x)) - D^2 f(x)) y_k^{\otimes 2} = D^3 f(\xi_k^1); \gamma_k b(x); y_k^{\otimes 2}$$

where $D^3 f(u); v; y^{\otimes 2} := \sum_{i,j} \langle \nabla D^2 f_{i,j}(u), v\rangle y_i y_j$ and $\xi_k^1 \in [x, x+\gamma_k b(x)]$. Thus, since $D^3 f$ is bounded, we deduce that

$$\Big|\mathbb{E}\Big\{ D^3 f(\bar{X}^W_{k-1}); \gamma_k b(\bar{X}^W_{k-1}); (\kappa(\bar{X}^W_{k-1})Q_k\Lambda_k)^{\otimes 2}/\mathcal{F}^W_{k-1}\Big\}\Big| \le C\gamma_k \int_{|y|\le u_k} |y|^2 \pi(dy) V^{\frac{a}{2}+r}(\bar{X}^W_{k-1}).$$

On the other hand, set $\Delta_\theta(\gamma_k, x) = \zeta_{k,\theta}(x) - (x + \gamma b(x))$. By Taylor's formula,

$$(D^2 f(\zeta_{k,\theta}(x)) - D^2 f(x+\gamma b(x))) y_k^{\otimes 2} = D^3 f(x); \Delta_\theta(\gamma_k, x); y_k^{\otimes 2} + \frac{1}{2} D^4 f(\xi_k^2); (\Delta_\theta(\gamma_k, x))^{\otimes 2}; y_k^{\otimes 2}$$

where $D^4 f(u); v^{\otimes 2}; y^{\otimes 2} = \sum_{i,j} D^2(D^2 f_{i,j}(u)) v^{\otimes 2} y_i y_j$ and $\xi_k^2 \in [x+\gamma b(x), \zeta_{k,\theta}(x)]$. The random variables U_k, $\bar{Z}_k$ and Λ_k are independent and independent of $\mathcal{F}^W_{k-1}$. Then, since $\mathbb{E}\{U_k/\mathcal{F}^W_{k-1}\} = \mathbb{E}\{\bar{Z}_k/\mathcal{F}^W_{k-1}\} = \mathbb{E}\{\Lambda_k^{\otimes 3}/\mathcal{F}^W_{k-1}\} = 0$, we have

$$\mathbb{E}\Big\{ D^3 f(\bar{X}^W_{k-1}); \Delta_\theta(\gamma_k, \bar{X}^W_{k-1}); (\kappa(\bar{X}^W_{k-1})Q_k\Lambda_k)^{\otimes 2}/\mathcal{F}^W_{k-1}\Big\} = 0.$$

Now, since $D^4 f$ is bounded, one checks that

$$\mathbb{E}\Big\{ |D^4 f(\bar{X}^W_{k-1}); \big(\Delta_\theta(\gamma_k, \bar{X}^W_{k-1})\big)^{\otimes 2}; (\kappa(\bar{X}^W_{k-1})Q_k\Lambda_k)^{\otimes 2}/\mathcal{F}^W_{k-1}|\Big\} \le C \int_{|y|\le u_k} |y|^2 \pi(dy) \gamma_k V^{2r}(\bar{X}^W_{k-1}).$$

Since $a/2 + r \le p/2 + a - 1$ and $2r \le p/2 + a - 1$, it follows that

$$|\mathbb{E}\{\tilde{R}_{k,2}(\gamma_k, \bar{X}^W_{k,2}, \bar{X}^W_{k-1}, Q_k\Lambda_k)/\mathcal{F}_{k-1}\}| \le C \int_{|y|\le u_k} |y|^2 \pi(dy) \gamma_k^2 V^{\frac{p}{2}+a-1}(\bar{X}^W_{k-1})$$

By a variant of Lemma $5(a).i$ and $(b).i$, we derive from the previous inequality that

$$\frac{1}{\sqrt{\Gamma_n \vee \Gamma_n^{(2)}}} \sum_{k=1}^{n} \mathbb{E}\{\tilde{R}_{k,2}(\gamma_k, \bar{X}^W_{k,2}, \bar{X}^W_{k-1}, Q_k\Lambda_k)/\mathcal{F}_{k-1}\} \xrightarrow[n\to+\infty]{\mathbb{P}} 0,$$

Then, assertion $c)$ follows from (37). □

6 An additional result

In this section, we present a partial extension when the Lévy process has a moment of order $2p$ with $p \in (1, 2]$. In this case, stating some global results as in Theorems 1 and 2 would need two kinds of restrictions: either to assume that at least the derivatives of f

tend to 0 when $|x| \to +\infty$ or to impose more constraints on the growth of the coefficients. The first alternative leads to a very technical proof and the second one can not be really envisaged for the drift term. Actually, we recall that in this type of problem, b produces the mean-reverting effect and then, it would not be natural to suppose that for instance, b is bounded .

That is why we propose to state a partial result for fast-decreasing steps for which the extension does only require some weak restrictions on f. We introduce a new assumption on the steps depending on the intensity of the mean-reverting:

$$\frac{\Gamma_n^{(2)}}{\sqrt{\Gamma_n}} \xrightarrow{n\to+\infty} 0 \quad \text{if } a=1 \quad \text{and,} \quad \frac{1}{\sqrt{\Gamma_n}}\sum_{k=1}^{n}\gamma_k^2\Gamma_k^{\frac{a\vee(2r)}{p}} \xrightarrow{n\to+\infty} 0 \quad \text{if } a<1. \tag{38}$$

Then,

Theorem 3. *Assume that $\mathbb{E}\{|Z_t|^{2p}\}<+\infty$ with $p\in(1,2]$ and that (2) admits a unique invariant measure ν . Let $a\in(0,1]$ and $r\geq 0$ such that $(\mathbf{R_a})$ and $(\mathbf{S_{a,r}})$ are satisfied and such that $p/2+a-1>r$. Suppose that $\mathbb{E}\{U_1^{\otimes 3}\}=0$, $\mathbb{E}\{|U_1|^4\}<+\infty$ and $\eta_n=\gamma_n$ for every $n\geq 1$. Let $f:\mathbb{R}^d\mapsto\mathbb{R}$ be a $\mathcal{C}^4$-function having bounded derivatives and satisfying $f(x)=O(\sqrt{V(x)})$ as $|x|\to+\infty$. Then,*
(a) Scheme (E): If (38) holds, $\sqrt{\Gamma_n}\bar{\nu}_n(Af) \underset{n\to+\infty}{\xrightarrow{\mathcal{L}}} \mathcal{N}\Big(0,\hat{\sigma}_f^2\Big)$.
(b) Scheme (P): If (38) holds and $\beta_{n,\pi}^{(2)}/\sqrt{\Gamma_n}\to 0$, the conclusion of (a) is valid for Scheme (P).
(c) Scheme (W): If (38) holds and $\beta_{n,\pi}^{(s)}/\sqrt{\Gamma_n}\to 0$, with $s=3$ or $s=4$ if π is quasi-symmetric in the neighborhood of 0, the conclusion of (a) is valid for Scheme (W).

Remark 10. We refer to [Pan06] for a proof of this result.
Assumption (38) is less constraining when $a=1$ because the L^p-control of the Euler scheme is better in this case (see Proposition 1). Note that when $a=1$, Theorem 3(a) corresponds to Theorem 1(a) when $\hat{\gamma}=0$.
Let $\gamma_k=\gamma_1 k^{-\zeta}$ with $\zeta\in(0,1]$ and $\gamma_1>0$. For Scheme (E), Theorem 3 applies in the following cases:

$$\zeta>\frac{1}{3} \quad \text{if } a=1 \quad \text{and} \quad \zeta>\frac{p+2\eta}{3p+2\eta} \quad \text{if } a<1$$

where $\eta=a\vee(2r)$. Since $\sqrt{\Gamma_n}\overset{\to+\infty}{\sim}\sqrt{\frac{\gamma_1}{1-\zeta}}n^{\frac{1-\zeta}{2}}$ if $\zeta\in(0,1)$, we derive that for every $\epsilon>0$, there exists an Euler scheme with polynomial step such that the rate of convergence is of order $n^{\frac{1}{3}-\epsilon}$ if $a=1$ and $n^{\frac{p}{3p+2\eta}-\epsilon}$ if $a<1$.

7 Numerical comparison of Schemes (P) and (W)

1. When $\nu(f)$ can be theoretically computed. In this first example, we are interested in the two-dimensional SDE

$$dX_t=-X_{t^-}dt+dZ_t \tag{39}$$

where Z is a symmetric purely discontinuous Lévy process (having no drift term). We consider $\phi: x\mapsto|x|^2$ and denote by ν, the unique invariant SDE (39). We can easily compute $\nu(\phi)$. In fact, as π is symmetric and $\nu(A\phi)=0$,

$$A\phi=-2\phi+\int|y|^2\pi(dy) \quad\Longrightarrow\quad \nu(\phi)=\frac{1}{2}\int|y|^2\pi(dy).$$

Let us test this theoretical result on a 2-dimensional example. Assume that

$$\pi^{(\alpha)}(dy) = 1_{\{|y|\leq 1\}}\frac{1}{|y|^{\alpha+2}}\lambda_2(dy) + 1_{\{|y|>1\}}\frac{1}{|y|^8}\lambda_2(dy) \quad \text{with } \alpha \in (1,3).$$

We have $\nu(\phi) = \pi(1/(2-\alpha)+1/4)$. In figures 3 and 7, we observe the rate for two values of α taking the choices of steps and truncation thresholds of Proposition 2(b). These

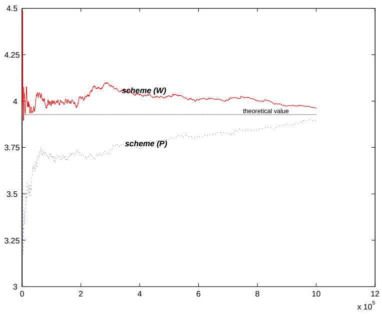

Figure 3: $n \mapsto \bar{\nu}_n(\phi)$, $\alpha = 1$

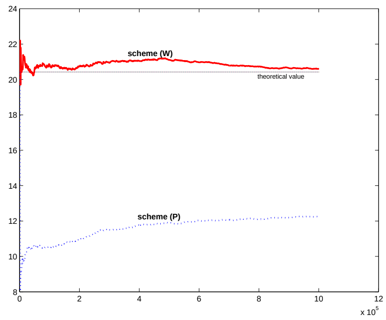

Figure 4: $n \mapsto \bar{\nu}_n(\phi)$, $\alpha = 5/3$

simulations are coherent with the theoretical results. Indeed, when $\alpha = 1$, the optimal asymptotic rates induced by Schemes (P) and (W) are the same (with order $n^{\frac{1}{3}}$). When $\alpha = 5/3$, the optimal asymptotic rate induced by scheme (W) is still of order $n^{\frac{1}{3}}$ whereas that of scheme (P) is of order $n^{\frac{1}{7}}$.

2. Another example. Now, we observe the following two-dimensional SDE

$$dX_t = -\frac{X_{t-}}{\sqrt{1+|X_{t-}|}}dt + (1+|X_{t-}|)^{\frac{1}{4}}dZ_t$$

where (Z_t) is a purely discontinuous Lévy process having no drift term with Lévy measure $\pi^{(\alpha)}$ (defined in the preceding example). One checks that Proposition 2 applies with $V(x) = 1+|x|^2$, $a = 3/4$, $r = 1/4$ and every $p \in (2,3)$. As in the preceding example, we test our procedure in the cases $\alpha = 1$ and $\alpha = 5/3$. As the dynamical system is less stable, the convergence is slower but we can observe the same phenomenon.

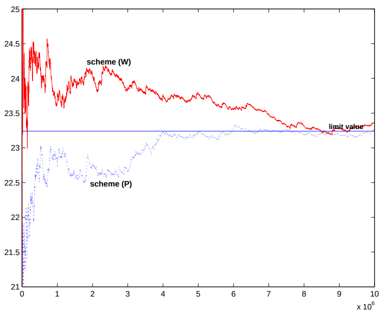

Figure 5: $n \mapsto \bar{\nu}_n(\phi)$, $\alpha = 1$

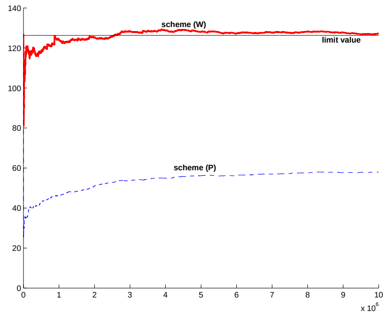

Figure 6: $n \mapsto \bar{\nu}_n(\phi)$, $\alpha = 5/3$

References

[AsRo01] Asmussen S., Rosinski J. (2001), Approximations of small jumps of Levy processes with a view towards simulation, J. of Applied Probability, 38, pp. 482-493.

[BaMiRe01] Barndorff-Nielsen O., Mikosh T., Resnick S. (2001), *Lévy Processes and Applications*, Birkauser.

[CoRo05] Cohen S., Rosinski J. (2005), Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered and operator stable processes, Preprint.

[EtKu86] Ethier S., Kurtz T. (1986), *Markov processes, characterization and convergence*, Wiley series in probability and mathematical statistics, Wiley, New York.

[HaHe80] Hall P., Heyde C. (1980), *Martingale Limit Theory and its Application*, Academic Press.

[LaPa02] Lamberton D., Pagès G. (2002), Recursive computation of the invariant distribution of a diffusion *Bernoulli*, 8, pp. 367-405.

[LaPa03] Lamberton D., Pagès G. (2003), Recursive computation of the invariant distribution of a diffusion: the case of a weakly mean reverting drift, *Stochastics and Dynamics*, 4, pp. 435-451.

[Lem05] Lemaire V. (2005), An adaptative scheme for the approximation of dissipative systems, *Stochastic Process. Appl.*, to appear.

[Lem06] Lemaire V. (2005), *Estimation numérique de la mesure invariante d'une diffusion*, Phd Thesis, http://tel.ccsd.cnrs.fr/tel-00011281.

[Pag01] Pagès G. (2001), Sur quelques algorithmes récursifs pour les probabilités numériques, *ESAIM : Probab. Statis.*, 5, pp. 141-170.

[Pan05] Panloup F. (2005), Recursive computation of the invariant measure of a stochastic differential equation driven by a Lévy process, http://hal.ccsd.cnrs//ccsd-00009273, Preprint.

[Pan06] Panloup F., Phd Thesis, Université Paris VI, in progress.

[PaVe01] Pardoux E., Veretennikov A. (2001), On Poisson equation and diffusion approximation I, *Ann. Probab.*, 29, pp. 1061-1085.

[PaVe03] Pardoux E., Veretennikov A. (2003), On Poisson equation and diffusion appoximation II, *Ann. Probab*, 31 , pp. 1166-1192.

[PaVe06] Pardoux E., Veretennikov A. (2006), On the Poisson equation and diffusion approximation III, to appear in *Ann. Probab.*